The Exponential Function

Basics

Consider the series of numbers a^1, a^2, a^3, … a^n, … where a is a positive number. It is clear that each number is a times the number before it. Conversely, each number is equal to the next number divided by a. It follows that, in order to maintain consistency, we must define a^0 to be a^1 / a. i.e. $a^0 = 1$, $a^{-1} = 1/a$, $a^{-2} = 1/a^2$ etc. etc.

We also have the relation $(a^n)^p = a^{np}$ which leads us to the idea that $\sqrt[p]{(a^{np})} = a^n$. i.e. the pth root of a is $a^{1/p}$.

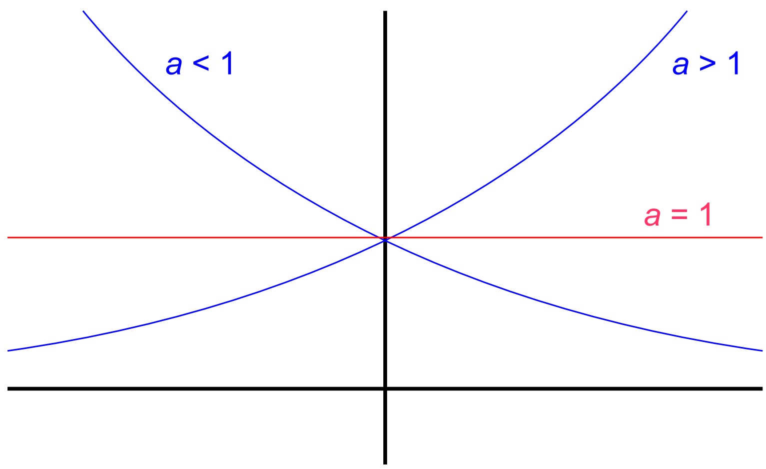
We now have a way of calculating the quantity a^x where a is positive and x is any number, positive, negative, integer or fractional.

Properties of the function y = a^x (a is positive)

Firstly, the function passes through the point (0 , 1) for all values of a.

Secondly, the function is always positive

Thirdly, if $a > 1$ the function increases monotonically from zero at $x = -\infty$ to ∞ at $x = \infty$. If $a < 1$, the function is the mirror image of the function generated by the reciprocal of a. If $a = 1$, y = 1 for all a.



Properties of the function y = (-a)^x

Since $(-a)^x = (-1)^x \,.\, a^x$ we must explore the function $y = (-1)^x$. Firstly we note that for all even integer values of x, the function equals one, and for all odd values, the function equals -1.

If $x = 0.5$, then our rules derived above insist that $(-1)^{0.5} = \sqrt{(-1)} = i$. In general, the result is a complex number whose modulus is 1 and whose argument is equal to πx. In other words, the function can be visualised as a helix rotating once round the x axis for every 2 units along the axis.

Likewise the function $y = (-a)^x$ can be visualised as a gradually expanding helix drawn on the surface of a huge bell or trumpet.

The relation between a^x and b^x

Suppose that $b = a^{b'}$ (where b' is simply some number related to b. For example, if $a = 2$ and $b = 8$, then b' would have to be equal to 3.)

Now $b^x = (a^{b'})^x = a^{b'.x}$ so in order to change the 'base' number of an exponential expression, you have to multiply (or divide) the exponent by a suitable factor b'. What is this factor? It is the number to which the 'base' number a must be raised to equal b and is called the 'logarithm of b to base a'. or $\log_a b$. We therefore have this important result

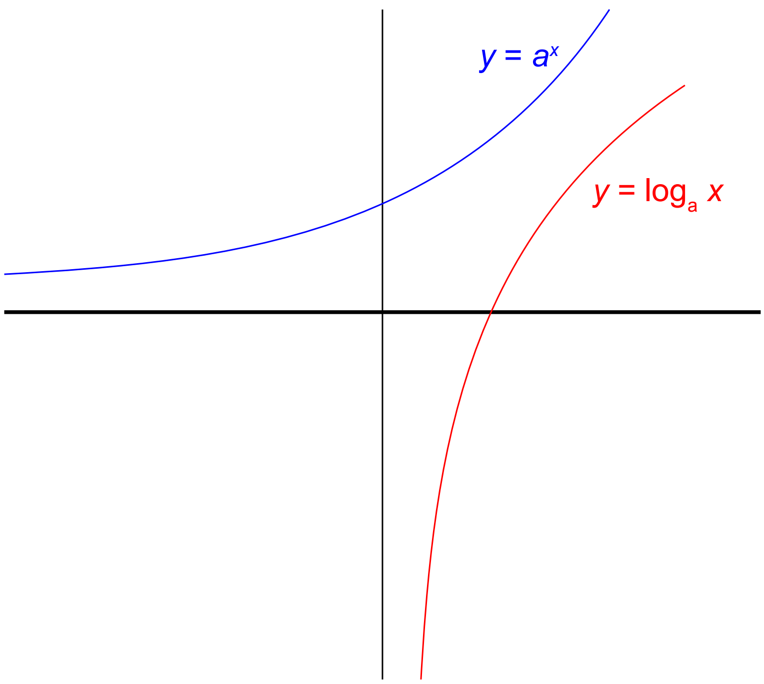
$$\boxed{b^x = a^{\log_a b \; . \; x}} \tag{1}$$

For example: 8^5 is equal to 2^{3*5} because the logarithm of 8 to base 2 is 3.

Now since $a^{\log_a b} = b$ we can see that taking the logarithm of a number (logarithmiation!) and exponentiation are inverse operations from which it follows that $\log_a(a^b) = b$

The logarithm function

Since exponentiation and logarithmiation are inverse functions, the latter is simply the former with the X and Y axes interchanged.



The relation between $\log_a x$ and $\log_b x$

Since $b^{\log_b x} = x$ $\qquad \log_a x = \log_a(b^{\log_b x})$

Using equation (1) $\qquad \log_a x = \log_a(a^{\log_a b . \log_b x})$

hence $\qquad \boxed{\log_a x = \log_a b . \log_b x} \qquad (2)$

For example, the log to base 8 of 32768 is 5 (because $8^5 = 32768$) The log to base 2 of this same number will be 15 because $\log_2 8 = 3$ and $3 * 5 = 15$.

Putting $x = a$ we see that
$$\log_a a \;=\; \log_a b \,.\, \log_b a \;=\; 1$$

hence
$$\boxed{\log_a b \;=\; \frac{1}{\log_b a}} \tag{3}$$

Two important identities

It is obvious that
$$\boxed{a^{(x_1+x_2)} \;=\; a^{x_1} \;\times\; a^{x_2}} \tag{4}$$

This can be summarised by saying that exponentiating a sum is the same as multiplying the individual exponents.

For example, if you wanted to add 3 and 5, you could exponentiate both of them to base 2 (getting 8 and 32 respectively); multiply these numbers together (getting 32768) and then take the logarithm of this number – 15! This would be a rather silly way to add two numbers together!

What is the equivalent expression in logarithms?

$$\log_a(x_1 . x_2) \;=\; \log_a(a^{\log_a x_1} . a^{\log_a x_2})$$

$$\log_a(x_1 . x_2) \;=\; \log_a(a^{\log_a x_1 \;+\; \log_a x_2})$$

$$\boxed{\log_a(x_1 . x_2) \;=\; \log_a x_1 \;+\; \log_a x_2} \tag{5}$$

In other words, the logarithm of a product is the same as adding the individual logarithms.

For example, if you wanted to multiply 8 by 32, you could take logarithms to base 2 (getting 3 and 5 respectively); add these numbers together (getting 15) and then exponentiate the result – 32768! This is a rather sensible way of multiplying large numbers (provided you have access to tables of logarithms and exponents, of course)

The gradient of a^x

The gradient of the function $y = a^x$ has a very interesting property.

$$\frac{\delta y}{\delta x} \;=\; \frac{a^{x+\delta x} - a^x}{\delta x} \;=\; a^x \frac{a^{\delta x} - 1}{\delta x}$$

As $\delta x \to 0$ the expression $\dfrac{a^{\delta x} - 1}{\delta x}$ will tend towards a certain value, say p. It follows that

$$\frac{dy}{dx} \;=\; p\, a^x \;=\; py$$

In other words, the gradient of the function is some constant p times the value of the function itself – and p will be the gradient of the function at $x = 0$ (because here the function has the value 1). We shall see how to calculate p from a in a minute.

The series expansion of a^x

Let us suppose that the function a^x can be expanded as follows:

$$y \;=\; a^x \;=\; a_0 \;+\; a_1 x \;+\; a_2 x^2 \;+\; a_3 x^3 + \ldots$$

We can differentiate this to get:

$$dy/dx = a_1 + 2a_2x + 3a_3x^2 + \dots$$

Now this has to be equal to p times y for all values of x. It follows therefore that $a_1 = pa_0$ and that $2a_2 = pa_1$ etc etc.

Now we know that $a_0 = 1$ (because $y = 1$ when $x = 0$) so we can calculate the values of all the coefficients as follows:

$$a_0 = 1 \;:\; a_1 = p \;:\; a_2 = \frac{1}{2}p^2 \;:\; a_3 = \frac{1}{2\,.\,3}p^3 \;:\; a_4 = \frac{1}{2\,.\,3\,.\,4}p^4 \dots$$

and we can write the complete expansion thus:

$$a^x = 1 + px + \frac{1}{2}p^2x^2 + \frac{1}{2\,.\,3}p^3x^3 + \frac{1}{2\,.\,3\,.\,4}p^4x^4 \dots$$

or

$$a^x = 1 + px + \frac{1}{2}(px)^2 + \frac{1}{3!}(px)^3 + \frac{1}{4!}(px)^4 \dots$$

The constant e

There is a value of a which has $p = 1$. This number is called e and it has the unique property that the gradient of the function e^x is equal to the value of the function itself at that point. We can calculate e simply by putting $p = 1$ and $x = 1$ in the above expression:

$$\boxed{e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} \dots = 2.7182818285 \dots} \qquad (6)$$

and

$$\boxed{e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 \dots} \qquad (7)$$

By using equation 1, we see that

$$a^x = e^{px} = 1 + px + \frac{1}{2}(px)^2 + \frac{1}{3!}(px)^3 + \frac{1}{4!}(px)^4 \dots$$

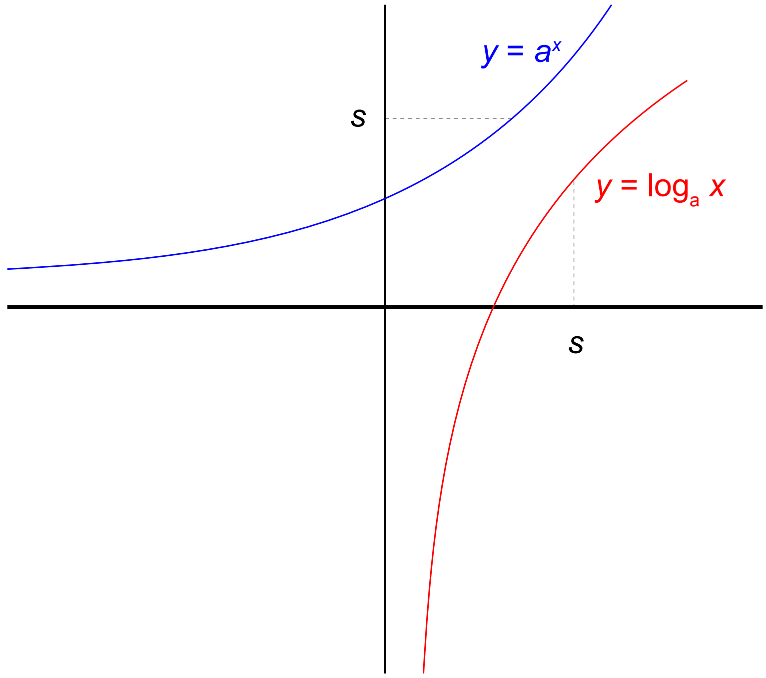
where $p = \log_e a$.

$\log_e a$ is called the *natural logarithm* of a and is written ln(a) or simply log(a)

The gradient of ln(x)

We have seen that the graph of $\log_a x$ is the inverse of the graph of a^x. This means that the gradient of $\log_a x$ at at point where $x = s$ is the reciprocal of the graph of a^x at the point where $y = s$. This is, of course, also true when $a = e$.

But the gradient of e^x at the point where $y = s$ is, of course, just s. So the gradient of ln(x) at the point where $x = s$ is exactly $1/s$. Conversely, we may say that the integral of $1/x$ is none other than the natural logarithm of x (plus a constant, of course).



This is a remarkable result. Normally, integrating the function x^{-1} is not possible using the usual rules for differentiation because the calculation involves dividing by zero. But here we have done it!

The series expansion of ln(x)

It is immediately clear than we cannot hope to find a polynomial expansion for $\ln(x)$ because the first term would have to be $-\infty$. It is, however possible to find a series expansion for the expression $\ln(x + 1)$ because this has the value 0 when $x = 0$ – which means that the first term of the expansion is also 0.

We can calculate the successive coefficients by differentiating the expression repeatedly and putting $x = 0$ each time (remembering to divide by the factorial as before)

	Expression	**Value at $x = 0$**	**Coefficient**
First term (a_0)	$y = \ln(x + 1)$	0	0
Second term (a_1)	$y' = (x + 1)^{-1}$	1	1
Third term (a_2)	$y'' = -(x + 1)^{-2}$	-1	-1/2! = -1/2
Fourth term (a_3)	$y''' = 2\,(x+1)^{-3}$	2	2/3! = 1/3
Fifth term (a_4)	$y'''' = -2.3\,(x+1)^{-4}$	-2.3	-2.3/4! = - 1/4
Sixth term (a_5)	$y''' = 2.3.4\,(x+1)^{-5}$	2.3.4	2.3.4/5! = 1/5

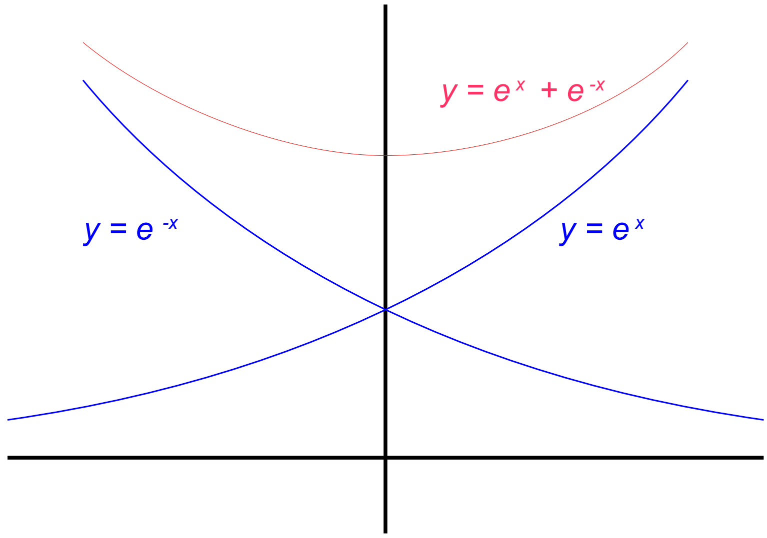
which leads us to the following important result

$$\boxed{\ln(x + 1) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \ldots} \qquad (9)$$

Of course, to be any use, this series must converge which it does in the range $[\,-1^+, 1\,]$ thus enabling us to calculate the natural logarithm of all numbers up to 2.

The hyperbolic trig functions

The graph of e^{-x} is, of course, the reflection of e^{x} in the Y axis. If we add them together we get a function which looks a bit like a hyperbola but which is in fact a completely new curve called a hyperbolic cosine or cosh(x). We shall see why in a minute.



The curve $y = e^{x} + e^{-x}$ actually passes through the point (0,2). In order to remove this slight anomaly, it is usual to divide the function by 2 so

$$\boxed{\cosh(x) = \frac{e^{x} + e^{-x}}{2}} \tag{10}$$

Looking back at the expansion of e^x (equation (7)) we see that all the odd terms will cancel leaving the following series expansion for cosh(x).

$$\boxed{\cosh(x) = 1 + \frac{1}{2}x^{2} + \frac{1}{4!}x^{4} + \frac{1}{6!}x^{6}\ldots} \tag{11}$$

Like wise we can define a function

$$\boxed{\sinh(x) = \frac{e^{x} - e^{-x}}{2}} \tag{12}$$

whose expansion is

$$\boxed{\sinh(x) = x + \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} + \frac{1}{7!}x^{7}\ldots} \tag{13}$$

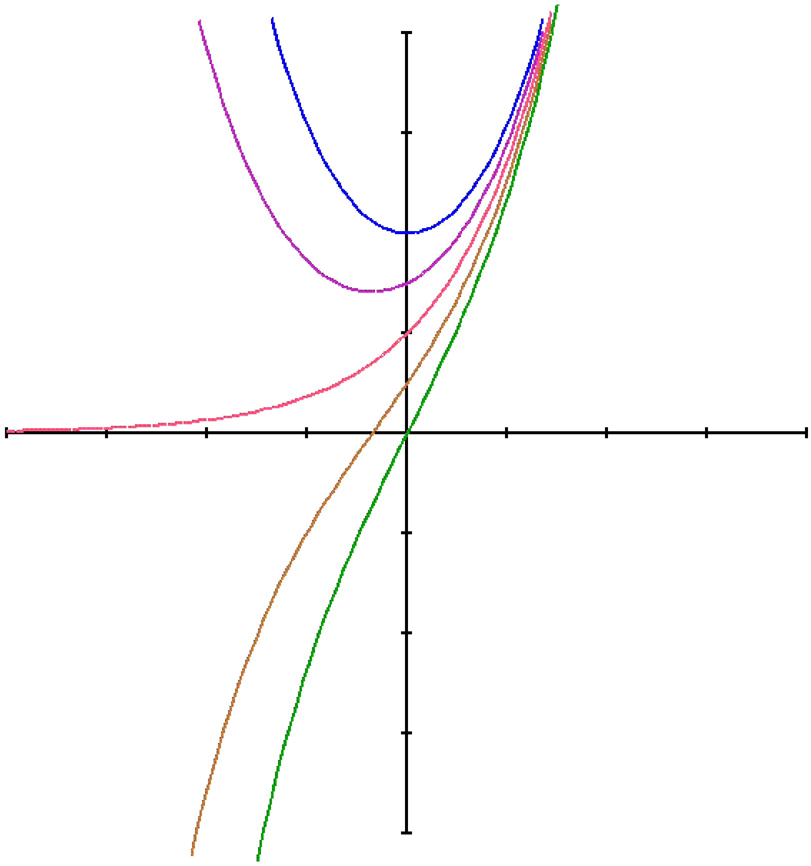
It is obvious that

$$\frac{\cosh(x) + \sinh(x)}{2} = e^{x}$$

Indeed the function

$$y = A\cosh(x) + (1 - A)\sinh(x)$$

(0 <= A <= 1) generates a whole family of curves which smoothly progress from the cosh(x) (blue) through e^x (red) to sinh(x) (green):



Differentiating the hyperbolic trig functions

It is easy to see that differentiating cosh(x) gives us sinh(x) and vice versa. Differentiating either function twice returns us to where we started. In other words, these functions form the solution to the differential equation

$$\frac{d^2 y}{dx^2} = ky$$

If x is interpreted as a time dimension, these functions therefore represent the possible trajectories of a particle whose acceleration at any time is proportional to its distance from the origin (and directed *away* from that origin). The cosh curve (blue) represents a particle which is thrown towards the origin but not sufficiently fast so that it is repelled before it gets there. The exponential curve (red) represents a particle which starts very close to the origin and gradually accelerates away from it. The sinh curve (green) represents a particle which is thrown towards the origin with enough speed to carry it right past.

The standard trig functions

Interesting though these functions are, as a physicist I am more interested in the solution to the equation

$$\frac{d^2 y}{dx^2} = -ky$$

This is, of course, the defining equation of oscillatory motion. What we need is a function (or pair of functions) which, when differentiated twice, turn into the negative of themselves – and therefore tune back into themselves after four steps). This gives us a big clue that what we are looking for must involve the number i. It is not too difficult to see that

$$\boxed{\cos(x) = \frac{e^{ix} + e^{-ix}}{2}} \tag{14}$$

and

$$\boxed{\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}} \tag{15}$$

fit the bill nicely provided that we simply assume that differentiating e^{ix} gives ie^{ix} etc, etc.

Moreover, we can easily show that

$$\boxed{\cos^2(x) + \sin^2(x) = 1} \tag{16}$$

and that

$$\boxed{\cos(x) + i\sin(x) = e^{ix}} \tag{17}$$

The series expansions of these functions are:

$$\boxed{\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \ldots} \tag{18}$$

and

$$\boxed{\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \ldots} \tag{19}$$

But are these functions the same as our familiar, geometrical sine and cosine functions? It is not by any means obvious, for example, that substituting $x = 2\pi$ or any multiple of that figure into equation (19) will generate zero; nor that putting equation (18) equal to equation (19) will generate the solution $x = \pi/4$ etc. but they do! They must!

It is worth putting $x = \pi$ into equation (18) just to see what happens. Here are the successive terms and the partial sums worked out on a spreadsheet:

0	2	4	6	8	10	12
1	-4.93	4.06	-1.34	0.24	-0.03	0
1	-3.93	0.12	-1.21	-0.98	-1	-1

The result rapidly converges on -1 – the cosine of π!

Since the factorial function increases faster than any polynomial, these series always converge for any value of x. On the other hand, as x gets larger, you need a lot of terms. For example, to work out the value of 4π you need at least 18 terms! In practice, of course, you only need to work out values up to $\pi/2$ which only needs about 6 terms to achieve sufficient accuracy.

It is really equation (16) that clinches the argument and proves that these curious exponential functions are, indeed, our familiar friends, the standard sine and cosine functions of the right angled triangle.

The value of π

There are many ways of calculating the value of π but none of them admit of an easy proof. The problem is that π only emerges as the *solution* to equations (18) and (19) for certain values of x.

We can, for example say that the solution to the equation

$$1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \ldots = 0$$

is $\pi/2$

If we take the first three terms and write $P = x^2$ we have

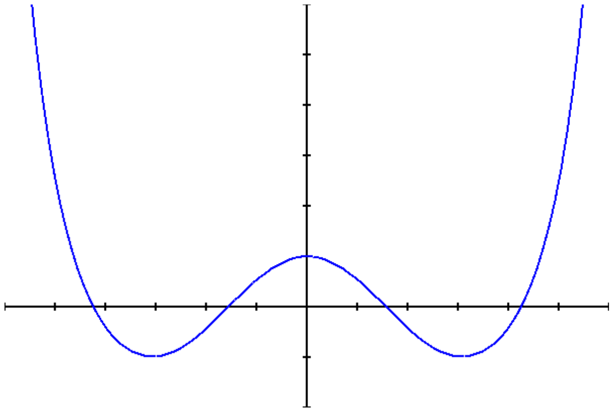
$$1 - \frac{1}{2}P + \frac{1}{24}P^2 = 0$$

the solution of which is $P = 2.535$ or 9.464 and $x = 1.592$ or 3.076

It is the first of these which we need giving us a value for π of 3.184. (The second number is actually an approximation to $3\pi/2$ but as such, it is practically useless)

We can get a more accurate value by using the fact that $\cos(\pi/4)$ is $1/\sqrt{2}$ from which we obtain $\pi = 3.143$

Here is a graph of the first 5 terms (up to x^8)



As each successive term is added the section near the origin approaches the shape of a cosine more and more closely and the crossing points on the axis get nearer and nearer to $\pi/2$ and $3\pi/2$. With more and more terms, the curve acquires more and more wiggles – but you need at least a dozen more terms to produce the next wiggle!

Complex Numbers

Polar coordinates

Equation (17) permits us to represent any complex number in two ways – effectively in either Cartesian or polar coordinates.

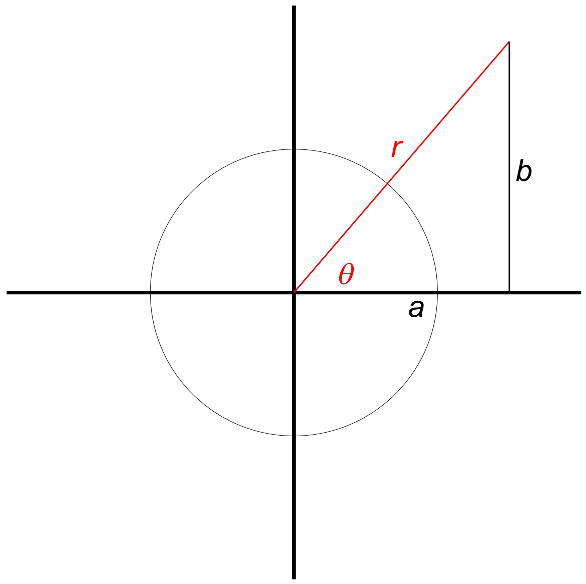
We can write

$$\boxed{z = a + ib = re^{i\theta}} \tag{20}$$

where the modulus $r = \sqrt{a^2 + b^2}$ and the argument $\theta = \arctan(b/a)$

or, alternatively, where $a = r\cos(\theta)$ and $b = r\sin(\theta)$

and we can plot complex numbers on the plane in the following way.



Adding and multiplying complex numbers

Adding complex numbers is easy – you simply add the real and the imaginary parts separately.

$$\boxed{(a + ib) + (c + id) = (a + c) + i(b + d)} \tag{21}$$

Basically what this means is that complex numbers add like ordinary vectors. This is the reason why complex numbers are useful in AC theory – phasors add in this way too.

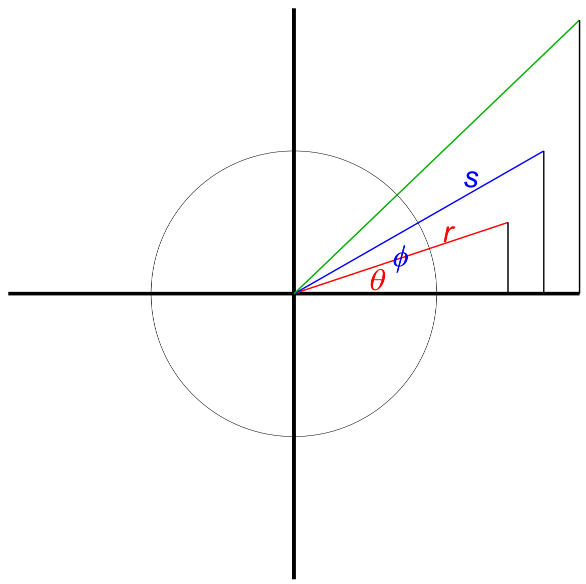
To multiply two complex numbers, you can use either their polar or their Cartesian forms.

$$\boxed{(a + ib) \times (c + id) = (ac - bd) + i(ad+bc)} \tag{22}$$

or

$$\boxed{re^{i\theta} \times se^{i\phi} = rse^{i(\theta + \phi)}} \tag{23}$$

It will be noted that the two moduli are multiplied together but the arguments are added. This is quite different from either the scalar or vector product of two vectors.

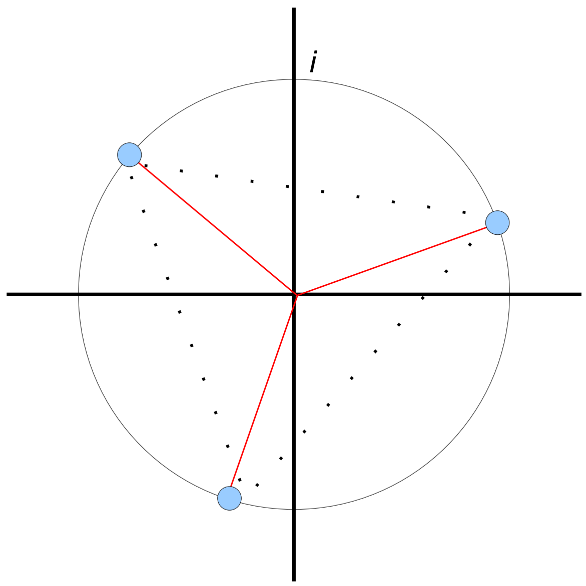


It is interesting to note that the following expressions follow immediately from this:

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$$
$$\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$$

The action of squaring a complex number simply squares the modulus and doubles the angle. Conversely, taking the square root of a complex number roots the modulus and halves the angle.

We must bear in mind, however, that any complex number with an argument of θ can equally well be represented with an argument of $(\theta + 2N\pi)$. When you take a root and divide the argument, some of these extra values will generate different multiple roots. The diagram below shows the cube roots of *i*.



Complex powers

We have seen how we can extend the simple idea that x^2 is $x \,.\, x$ to negative and fractional exponents; but what about imaginary exponents?

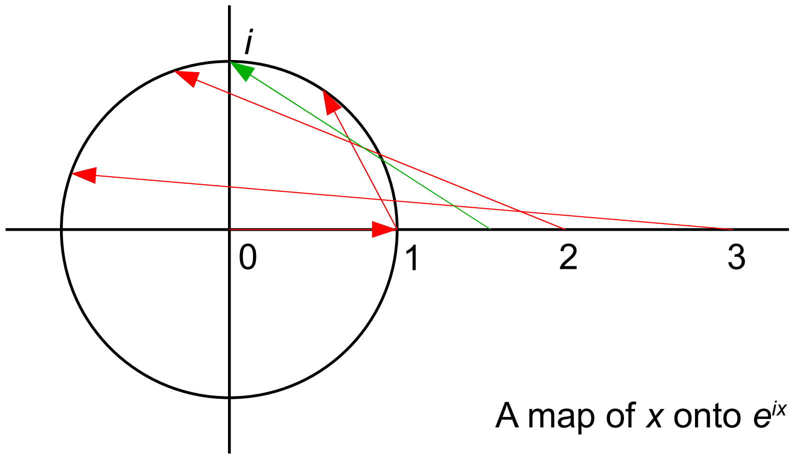
Let us take equations (18) and (19) as our definitions of sin and cos – after all, they do not contain any imaginary numbers. We can also take equation (17) to be the definition of e^{ix} – but if we do, we must check that it differentiates correctly:

$$y = e^{ix} = \cos(x) + i\sin(x)$$

$$\frac{dy}{dx} = -\sin(x)+i\cos(x) = i(i\sin(x)+\cos(x)) = i\,e^{ix}$$

so that is OK.

What this means so far is that if we start with the number e and raise it to the power i we get the complex number (0.5403 + i 0.8414). (I have used a calculator to calculate cos(1) and sin(1) but I should really have used equations (18) and (19). Maybe the calculator uses these equations anyway!). Because of equation (16) we see that this number must lie on the unit circle. Indeed, all numbers of the form e^{ix} lie on the unit circle and x is the argument of the number (the angle at the origin in radians).



A map of x onto e^{ix}

What about a^{ix} ?

From equation (1) $$a^{ix} = e^{\ln(a).ix} = e^{i.\ln(a).x}$$

In other words, all real numbers, when raised to an imaginary number are mapped onto the unit circle.

What about $e^{x + iy}$?

That's easy. $$e^{x+iy} = e^x.e^{iy} = e^x(\cos(y) + i\sin(y))$$

What this means is that if you raise a real number to a complex power, you get a complex number whose argument (y) is the imaginary part and whose modulus is e to the power of the real part (x).

What about $(a + ib)^{(x+iy)}$?

This is more difficult. Lets have a go at i^i first.

Suppose that we find a (real) number x such that $e^{ix} = i$

Raising to the power i *gives* $i^i = (e^{ix})^i = e^{-x}$

A glance at the diagram above will reveal that one possible value of x (indicated by the green arrow) is π/2. (You can add or subtract any multiple of 2π to this to find other values.)

This means that the principle value of i^i is $e^{-\pi/2}$ which is, rather surprisingly, a real number! It has the value 0.2079...

The general case of w^z

Since there is no general algebraic way of expanding an expression like $(a + b)^x$ it is better if we express w in polar coordinates as $re^{i\theta}$

$$w^z = \left(re^{i\theta}\right)^{(x + iy)} = r^{(x + iy)}e^{i\theta(x+iy)} = r^x r^{iy} e^{-y\theta} e^{i\theta x}$$

which, of course, is just another complex number. What is surprising is that there is no need to introduce any new 'hyper-complex' numbers to describe the square root of i for example, or the i^{th} root of π. They all have answers within the complex domain.

Lets check the case of i^i . Here $r = 1$, $\theta = \pi/2$ ($\pm 2N\pi$) , $x = 0$ and $y = 1$

$$i^i = 1^0 1^i e^{-\pi/2} e^0 = e^{-\pi/2}$$

(The i^{th} root of π is $\pi^{1/i} = \pi^{-i} = e^{-i\log\pi}$ i.e. a point on the unit circle about 65^0 below the real axis.)

The logarithm of a complex number.

If we represent a complex number in polar coordinates eg $z = re^{i\theta}$ we can define

$$\boxed{\log(z) = \log(r) + i\theta} \tag{24}$$

(Exponentiating both sides of this equation will quickly verify the result)

It should be born in mind that you can add or subtract 2π any number of times from θ so a more complete expression is

$$\log(z) = \log(r) + i(\theta + 2N\pi)$$