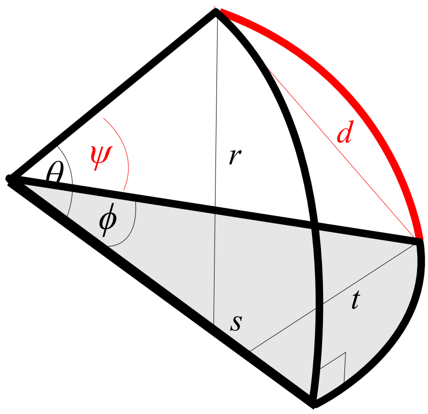
Spherical triangles

Right angled triangle

Consider a right angled spherical triangle with arcs θ and ϕ.



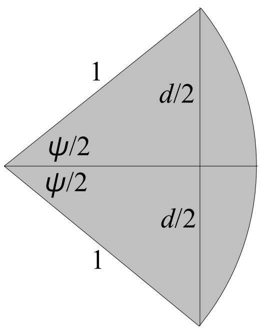
In order to calculate the length of the (thin) red line we need the following:

$$r = \sin\theta$$
$$t = \sin\phi$$
$$s = \cos\phi - \cos\theta$$

By Pythagoras

$$d^2 = \sin^2\theta + (\cos\phi - \cos\theta)^2 + \sin^2\phi$$
$$d^2 = 2 - 2\cos\theta\cos\phi$$

To work out the angular length of this chord we need the following result:



$$\sin\psi/2 = d/2$$
$$\cos\psi/2 = \sqrt{1 - d^2/4}$$
$$\sin\psi = 2\ \sin\psi/2\ \cos\psi/2 = d\sqrt{1 - d^2/4}$$

from which we get:

$$\begin{aligned}\sin\psi &= \sqrt{2 - 2\cos\theta\cos\phi}\ \sqrt{1 - 1/2 + 1/2\cos\theta\cos\phi} \\ &= \sqrt{(1 - \cos\theta\cos\phi)(1 + \cos\theta\cos\phi)}\end{aligned}$$

hence

$$\boxed{\sin\psi = \sqrt{1 - \cos^2\theta\cos^2\phi}}$$

Alternatively we can write:

$$\begin{aligned}\cos\psi &= \cos^2\psi/2 - \sin^2\psi/2 \\ &= (1 - d^2/4) - d^2/4 \\ &= 1 - d^2/2\end{aligned}$$

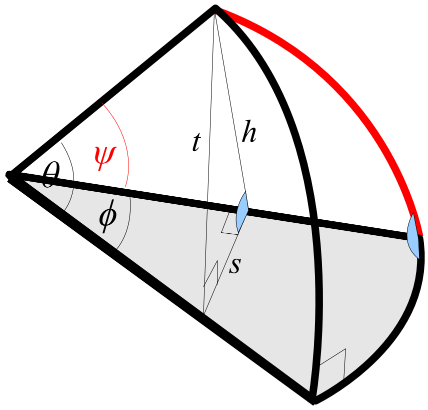
hence

$$\boxed{\cos\psi = \cos\theta\cos\phi}$$

Now this formula is the spherical equivalent of Pythagoras' theorem to which it must reduce when θ and ϕ are small – which it does:

$$\begin{aligned}(1 + \psi^2 + ...) &= (1 + \theta^2 + ...)(1 + \phi^2 + ...) \\ \psi^2 &= \theta^2 + \phi^2\end{aligned}$$

The question now arises, what are the other two angles? One of them (let us call it *β*) is marked in blue in the diagram below. It is the angle between the horizontal plane and the inclined plane.



Now, although it does not look like it from this perspective, the triangle with side *s*, *t* and *h* is a right angled triangle with *h* as the hypotenuse. It is easy to see that:

$$\begin{aligned}h &= \sin\psi \\ t &= \sin\theta \\ s &= \cos\psi\tan\phi\end{aligned}$$

We therefore deduce that

$$\sin\beta = \frac{\sin\theta}{\sin\psi}$$

$$\cos\beta = \frac{\cos\psi\tan\phi}{\sin\psi} = \frac{\tan\phi}{\tan\psi}$$

$$\tan\beta = \frac{\sin\theta}{\cos\psi\tan\phi}$$

Of these, the best formula to use is the last because we can substitute for cos *ψ* to obtain:

$$\tan\beta = \frac{\sin\theta}{\cos\theta\cos\phi\tan\phi}$$

$$\boxed{\tan\beta = \frac{\tan\theta}{\sin\phi}}$$

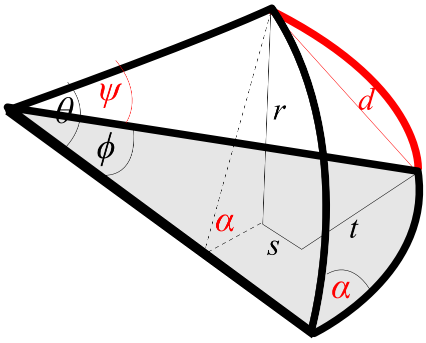
and, of course:

$$\boxed{\tan\gamma = \frac{\tan\phi}{\sin\theta}}$$

(where γ is the third angle)

The general triangle

So far we have considered a right angled triangle with short sides θ and φ with hypotenuse ψ. The three angles we shall call α, β and γ with α opposite ψ, β opposite θ and γ opposite φ.



Now we have

$$r = \sin\theta\sin\alpha$$
$$t = \sin\phi - \sin\theta\cos\alpha$$
$$s = \cos\phi - \cos\theta$$

hence

$$\begin{aligned} d^2 &= \sin^2\theta\sin^2\alpha + (\cos\phi - \cos\theta)^2 + (\sin\phi - \sin\theta\cos\alpha)^2 \\ &= \sin^2\theta\sin^2\alpha + \cos^2\phi - 2\cos\phi\cos\theta + \cos^2\theta + \sin^2\phi - 2\sin\phi\sin\theta\cos\alpha + \sin^2\theta\cos^2\alpha \\ &= \sin^2\theta(\sin^2\alpha+\cos^2\alpha) + \cos^2\theta + (\sin^2\phi+\cos^2\phi) - 2(\cos\phi\cos\theta+\sin\phi\sin\theta\sin\alpha) \\ &= 2 - 2(\cos\theta\cos\phi + \sin\phi\sin\theta\cos\alpha) \end{aligned}$$

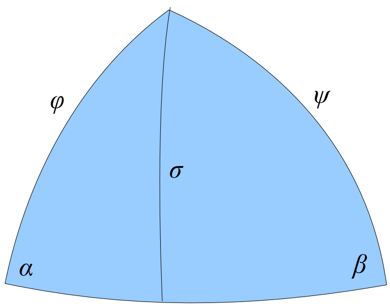
Since

$$\cos\psi = 1 - d^2/2$$

we have

$$\boxed{\cos\psi = \cos\theta\cos\phi + \sin\theta\sin\phi\cos\alpha}$$

which is the cosine rule for spherical triangles!



In the general triangle shown above we have:

$$\sin\alpha = \frac{\sin\sigma}{\sin\phi} \quad \text{and} \quad \sin\beta = \frac{\sin\sigma}{\sin\psi}$$

from which we can deduce the 'sine' rule for spherical triangles:

$$\boxed{\frac{\sin\alpha}{\sin\psi} = \frac{\sin\beta}{\sin\phi} \left[= \frac{\sin\gamma}{\sin\theta}\right]}$$

Longitude and Latitude

Our system of defining the position of places on the surface of the Earth is not ideal, mathematically speaking because lines of latitude are not great circles.

A system of Longitude and 'Langitude' (ie great circles based on an East/West axis) would have the desirable property that distances could be calculated using the spherical triangle formulae derived above but 'Langitude' would have no geographical significance and, even worse, places on the great circle which passes through all four poles would have no unique reference.

A system of 'Lotitude' (ie distance from the prime meridian) and Latitude has the disadvantage that every set of coordinates describes two points on the Earth's surface and that many coordinate sets do not exist at all.

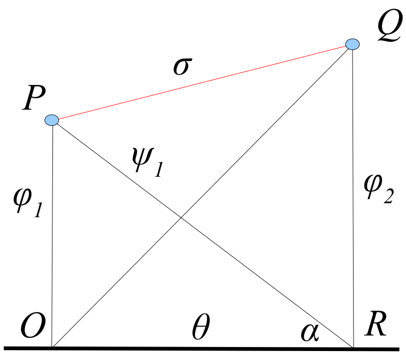
So we are stuck with a system which has great geographical significance but which treats the two coordinates differently.

The distance between two points on a sphere

What is the distance between the points (θ_1, ϕ_1) and (θ_2, ϕ_2) (where θ is longitude and ϕ is latitude)?

It is clear that the distance is the same as the distance between the points $(0, \phi_1)$ and (θ, ϕ_2) where $\theta = \theta_2 - \theta_1$

We can draw the situation in the following way provided that we understand that all the lines are great circles and that the thick line is the equator.



In the triangle *OPR* we have:

$$\cos\psi_1 = \cos\theta\cos\phi_1$$
$$\sin\alpha = \frac{\sin\phi_1}{\sin\psi_1}$$

In the triangle *PQR* we have:

$$\begin{aligned}\cos\sigma &= \cos\psi_1\cos\phi_2 + \sin\psi_1\sin\phi_2\cos(90 - \alpha)\\ &= \cos\psi_1\cos\phi_2 + \sin\phi_1\sin\phi_2\\ &= \cos\theta\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2\end{aligned}$$

which has the expected symmetry.

For example, what is the distance between London (51^0 30' N, 0^0) and New York (40^0 40' N, 74^0 E)?

$$\cos\sigma = \cos(74)\cos(51.5)\cos(40.7) + \sin(51.5)\sin(40.7) = 0.130 + 0.510 = 0.640$$
$$\sigma = 50.2^0$$
$$d = 6400 \times 50.2 \times \pi/180 = 5600\,km$$