Regular solids

Platonic solids

These are solids made of one kind of regular polygon only. There are 5 of them

Notation

N = number of vertices on each face
P = number of faces meeting at each vertex
F = number of faces
V = number of vertices
E = Number of edges

Theorem 1.1

$$\boxed{F = 4P/Z \quad \text{where} \quad Z = 2N - P(N-2)}$$

Euler's theorem $F + V = E + 2$

also $V = F \times N / P$

and $E = F \times N / 2$

so $F + F\,N/P - F\,N/2 = 2$

hence $F = 2/(1 + N/P - N/2) = 4\mathrm{P}/(2\mathrm{P} + 2\mathrm{N} - NP) = 4\mathrm{P}/Z$

and $V = 4\,N/Z$

where $Z = 2P + 2N - NP = 2N - P(N-2)$

For example, a dodecahedron has $N = 5$ and $P = 3$ so $Z = 6 + 10 - 15 = 1$ ie: $F = 4 * 3 / 1 = 12$ faces

An alternative proof using vertex deficit

Internal angle of an N-sided polygon $= \pi - 2\pi/N$

Vertex deficit $\psi = 2\pi - P(\pi - 2\pi/N) = \pi(2 - P + 2P/N) = \pi Z/N$

where $Z = 2N - P(N-2)$

Now by the vertex deficit theorem (see Solid angles and Polyhedra)

Number of vertices $V = 4\pi/\psi = 4\pi/(\pi Z/N) = 4N/Z$

Since $V = F \times N/P$

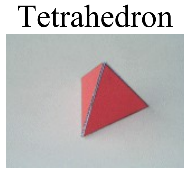
$F = 4P/Z$

It is obvious that Z must be greater than zero and must not contain any prime factors (other than 2) which are incompatible with either N or P.

A graph of Z against N and P has the following form:

***P* = 6**	8	4	0	-4	-8	-12
***P* = 5**	7	4	1	-2	-5	-8
***P* = 4**	6	4	2	0	-2	-4
***P* = 3**	5	4	3	2	1	0
***P* = 2**	4	4	4	4	4	4
***P* = 1**	3	4	5	6	7	8
	***N* = 1**	***N* = 2**	***N* = 3**	***N* = 4**	***N* = 5**	***N* = 6**

Obviously no solids exist with $N < 3$, $P < 3$ or $Z < 0$. These squares are shaded in grey. If $Z = 0$, the number of faces is infinite. This happens with $N = 3$ and $P = 6$ which is a plane tessellated with triangles. Similarly $N = 4$ and $P = 4$ produces a plane tessellated with squares and $N = 6$, $P = 3$ tessellates the plane with hexagons. The only possibilities left all generate regular polyhedra as follows:

***P* = 5**	F = 20 Icosahedron  Triball		
***P* = 4**	F = 8 Octahedron  Diamond		
***P* = 3**	F = 4 Tetrahedron  Pyramid	F = 6 Hexahedron  Cube	F = 12 Dodecahedron  Pentaball
	***N* = 3**	***N* = 4**	***N* = 5**

Complimentary solids

For any regular solid it is possible to construct a complimentary solid by replacing each vertex with a face and each face with a vertex. In the case of the Platonic solids the cube becomes the octahedron (and vice versa); the pentaball becomes the icosahedron (and vice versa) while the tetrahedron is its own complement.

Since *F* and *V* are interchanged, Euler's theorem tells us that complimentary solides will always have the same n umber of edges *E*.

Archimedean solids

It is possible to make an infinite number of prisms and drums but if we leave these out, we find that there are 10 solids which comprise 2 types of polygon, and 3 which contain three types.

Archimedean solids containing 2 types of polygon

Notation:

N_A = number of vertices on each face A
N_B = number of vertices on each face B
P_A = number of faces A meeting at each vertex
P_B = number of faces B meeting at each vertex
F_A = number of faces A
F_B = number of faces B
V = number of vertices
E = Number of edges

Theorem 1.2

$$\boxed{\begin{gathered} F_A = 4\mathrm{P}_A N_B / Z \quad : \quad F_B = 4\mathrm{P}_B N_A / Z \\ Z = 2N_A N_B - P_A N_B(N_A-2) - P_B N_A(N_B-2) \end{gathered}}$$

Euler's theorem $\quad F_A + F_B + V - E = 2$

also $\quad V = (F_A N_A + F_B N_B)/(P_A + P_B)$

and $\quad E = (F_A N_A + F_B N_B)/2$

so $\quad F_A + F_B + (F_A N_A + F_B N_B) \times (1/(P_A + P_B) - 1/2) = 2$

$$F_A + F_B - (P_A + P_B - 2)(F_A N_A + F_B N_B) \; / \; 2(P_A + P_B) = 2$$

But $\quad (V =) F_A N_A / P_A = F_B N_B / P_B$

From which $\quad F_A = 4P_A N_B / Z \quad$ and $\quad F_B = 4P_B N_A / Z$

where $\quad Z = 2P_A N_B + 2P_B N_A - N_A N_B (P_A + P_B - 2)$

$$= 2N_A N_B - P_A N_B (N_A - 2) - P_B N_A (N_B - 2)$$

For example, the solid which has two opposite squares joined by 8 triangles (ie a square drum) has

$N_A = 3$, $N_B = 4$ and $P_A = 3$, $P_B = 1$ so $Z = 24 + 6 - 24 = 6$

ie: $F_A = 4*3*4/6 = 8$ and $F_B = 4*3*1/6 = 2$

It is interesting to note that when $N_A = 3$, and $P_A = 3$, $P_B = 1$, Z is always equal to 6 regardless of N_B and therefore F_B is always equal to 2. This represents an infinite series of solids consisting of two parallel polygons separated by a ring of triangles. These are called antiprisms (I prefer *drums*).

It is also interesting to note that when $N_A = 4$, and $P_A = 2$, $P_B = 1$, Z is always equal to 8 regardless of N_B and therefore F_B is always equal to 2. This represents an infinite series of solids consisting of two parallel polygons separated by a ring of squares. These are called prisms

In general, for any given values of N_A and N_B, the value of Z will be positive only for small values of P_A and P_B . If Z is negative, you are trying to fit too many polygons round each point. If Z = 0, the vertex is flat and the situation may result in a plane tessellation.

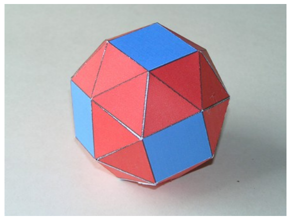
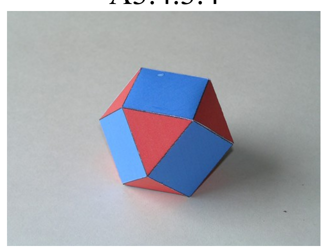
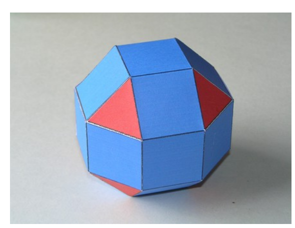
It is worth noting that the existence of a solution with integral values of Z, F_A, F_B and V does not necessarily imply that the solid actually exists.

- *Consider those solids with triangles and squares ie* $N_A = 3$ *and* $N_B = 4$

A graph of Z against P_A and P_B has the following form:

$\boldsymbol{P_A = 6}$	-4	-12	-18	-24
$\boldsymbol{P_A = 5}$	-2	-8	-14	-20
$\boldsymbol{P_A = 4}$	2	-4	-10	-16
$\boldsymbol{P_A = 3}$	6	0	-6	-12
$\boldsymbol{P_A = 2}$	10	4	-2	-8
$\boldsymbol{P_A = 1}$	14	8	2	-4
	$\boldsymbol{P_B = 1}$	$\boldsymbol{P_B = 2}$	$\boldsymbol{P_B = 3}$	$\boldsymbol{P_B = 4}$

Obviously no solids exist with $Z < 0$. These squares are shaded in grey. If $Z = 0$, the number of faces is infinite. This happens with $P_A = 3$ and $P_B = 2$ which is a plane tessellated with 3 triangles and 2 squares round each vertex. (There are two ways of doing this actually.) Of the seven possibilities left only five generate integer values for F_A and F_B and hence produce solids as follows:

***P*$_A$ = 4**	$F_A = 32$, $F_B = 6$ A3-3-3-3-4  Dilated Cube		
***P*$_A$ = 3**	$F_A = 8$, $F_B = 2$ D3.3.3.4 Square Drum		
***P*$_A$ = 2**	Does not exist	$F_A = 8$, $F_B = 6$ A3.4.3.4  Chopped Diamond	
***P*$_A$ = 1**	Does not exist	$F_A = 2$, $F_B = 3$ P3.4.4 Triangular Prism	$F_A = 8$, $F_B = 18$ A3.4.4.4  Exploded Cube
	***P*$_B$ = 1**	***P*$_B$ = 2**	***P*$_B$ = 3**

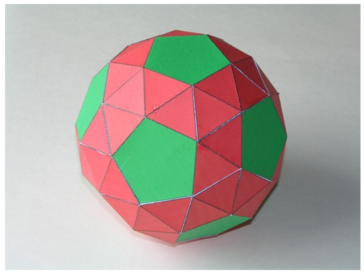
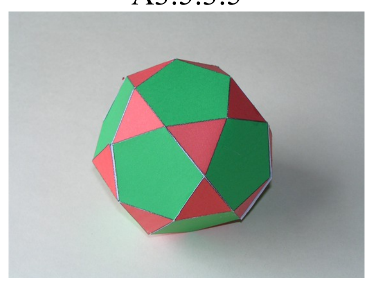
- *Consider those solids with triangles and pentagons ie $N_A = 3$ and $N_B = 5$*

A graph of Z against P_A and P_B has the following form:

***P*$_A$ = 6**	-9	-18	-27	-36
***P*$_A$ = 5**	-4	-13	-22	-31
***P*$_A$ = 4**	1	-8	-17	-26
***P*$_A$ = 3**	6	-3	-12	-21
***P*$_A$ = 2**	11	2	-7	-16
***P*$_A$ = 1**	16	7	-2	-11

	$P_B = 1$	$P_B = 2$	$P_B = 3$	$P_B = 4$

Obviously no solids exist with $Z < 0$. These squares are shaded in grey.

$P_A = 4$	$F_A = 40$, $F_B = 12$ A3.3.3.3.5  Dilated Pentaball	
$P_A = 3$	$F_A = 10$, $F_B = 2$ D3.3.3.5 Pentagonal Drum	
$P_A = 2$	Does not exist	$F_A = 20$, $F_B = 12$ A3.5.3.5  Chopped Pentaball
$P_A = 1$	Does not exist	Does not exist
	$P_B = 1$	$P_B = 2$

- *Consider those solids with triangles and hexagons ie $N_A = 3$ and $N_B = 6$*

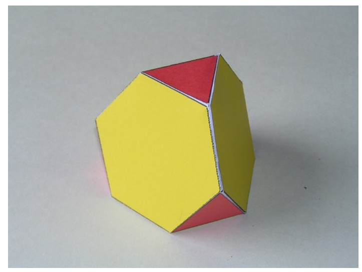
A graph of Z against P_A and P_B has the following form:

$P_A = 5$	-6	-18	-30
$P_A = 4$	0	-12	-24
$P_A = 3$	6	-6	-18
$P_A = 2$	12	0	-12
$P_A = 1$	18	6	-6
	$P_B = 1$	$P_B = 2$	$P_B = 3$

Obviously no solids exist with $Z < 0$. These squares are shaded in grey. When Z = 0 the result is a tessellated plane.

When Z = 18, the number of faces is fractional so the solid does not exist. When Z = 12, the number of faces is integral but the solid is impossible to build without bending the hexagon!

That leaves just two possibilities.

$P_A = 3$	$F_A = 12$, $F_B = 2$ D3.3.3.6 Hexagonal Drum	
$P_A = 2$	$F_A = 4$, $F_B = 1$ Impossible	
$P_A = 1$	Does not exist	$F_A = 4$, $F_B = 4$ A3.6.6  Clipped Pyramid
	$P_B = 1$	**$P_B = 2$**

- *Consider those solids with triangles and heptagons ie $N_A = 3$ and $N_B = 7$*

A graph of Z against P_A and P_B has the following form:

$P_A = 5$	-8	-23	-48
$P_A = 4$	-1	-16	-41
$P_A = 3$	6	-9	-34
$P_A = 2$	13	-2	-27
$P_A = 1$	20	5	-20
	$P_B = 1$	**$P_B = 2$**	**$P_B = 3$**

The only one of these that works is $P_A = 3$, $P_B =1$ which is the heptagonal antiprism

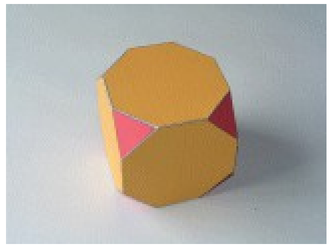
Consider those solids with triangles and octagons ie $N_A = 3$ and $N_B = 8$

A graph of Z against P_A and P_B has the following form:

$P_A = 5$	-10	-28	-46
$P_A = 4$	-2	-20	-38
$P_A = 3$	6	-12	-30
$P_A = 2$	14	-4	-22
$P_A = 1$	22	4	-14
	$P_B = 1$	**$P_B = 2$**	**$P_B = 3$**

Of the cases where Z is positive, $P_A = 3$, $P_B =1$ is the octagonal antiprism.

The case $P_A = 1$, $P_B = 2$ has vertex number A3.8.8 and has 8 triangles and 6 octagons and is a cube with its corners clipped off.



Clipped Cube

- *Consider those solids with triangles and nonagons ie $N_A = 3$ and $N_B = 9$*

A graph of Z against P_A and P_B has the following form:

$P_A = 5$	-12	-33	-54
$P_A = 4$	-3	-24	-45
$P_A = 3$	6	-15	-36
$P_A = 2$	15	-6	-27
$P_A = 1$	24	3	-18
	$P_B = 1$	$P_B = 2$	$P_B = 3$

Of the cases where Z is positive, $P_A = 3$, $P_B = 1$ is the nonagonal antiprism.

The case $P_A = 1$, $P_B = 2$ has an integral number of faces and vertices but is impossible to construct.

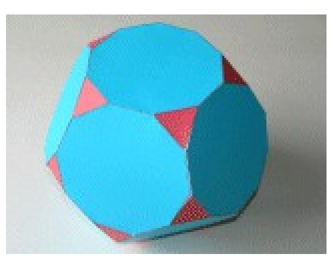
- *Consider those solids with triangles and decagons ie $N_A = 3$ and $N_B = 10$*

A graph of Z against P_A and P_B has the following form:

$P_A = 5$	-14	-38	-62
$P_A = 4$	-4	-28	-52
$P_A = 3$	6	-18	-42
$P_A = 2$	16	-8	-32
$P_A = 1$	26	2	-22
	$P_B = 1$	$P_B = 2$	$P_B = 3$

Of the cases where Z is positive, $P_A = 3$, $P_B = 1$ is the decagonal antiprism.

The case $P_A = 1$, $P_B = 2$ has vertex numbers 3.10.10. It has 20 triangles and 12 decagons and is a dodecahedron with its corners clipped off.



Clipped Pentaball

- *Consider those solids with triangles and larger polygons ie* $N_A = 3$ *and* $N_B > 10$

Only the antiprisms exist.

- *Consider those solids with squares and pentagons ie* $N_A = 4$ *and* $N_B = 5$

A graph of Z against P_A and P_B has the following form:

$P_A = 4$	-12	-24	-36
$P_A = 3$	-2	-14	-26
$P_A = 2$	8	-4	-16
$P_A = 1$	18	6	-6
	$P_B = 1$	$P_B = 2$	$P_B = 3$

There is only one possibility, the pentagonal prism.

- *Consider those solids with squares and hexagons ie* $N_A = 4$ *and* $N_B = 6$

A graph of Z against P_A and P_B has the following form:

$P_A = 4$	-16	-32	-48
$P_A = 3$	-4	-20	-36
$P_A = 2$	8	-8	-24
$P_A = 1$	20	4	-12
	$P_B = 1$	$P_B = 2$	$P_B = 3$

There are two possibilities, the hexagonal prism and (4.4.6) the clipped octahedron (4.6.6)

Clipped Diamond

- *Consider those solids with squares and heptagons ie* $N_A = 4$ *and* $N_B = 7$

There is only one possibility, the heptagonal prism.

- *Consider those solids with squares and octagons ie* $N_A = 4$ *and* $N_B = 8$

Again there is only one possibility, the octagonal prism

This is true for all combinations of squares and larger polygons

- *Consider those solids with pentagons and hexagons ie* $N_A = 5$ *and* $N_B = 6$

A graph of Z against P_A and P_B has the following form:

$P_A = 4$	-32	-52	-72
$P_A = 3$	-14	-34	-54
$P_A = 2$	4	-16	-36
$P_A = 1$	22	2	-18
	$P_B = 1$	$P_B = 2$	$P_B = 3$

There are two possibilities: 5.5.6 and 5.6.6. The first does not work. The second is the familiar Buckyball with 12 pentagons and 20 hexagons

Clipped Triball or Buckyball

Complementary solids of Archimedean solids (type 2)

In order to make a complimentary solid you must construct a plane at each vertex which is normal to the line joining the vertex to the centre of the original solid. Since every vertex of an Archimedean solid is identical, it follows that every face of its complement will also be identical. But is does not follow that these faces will be regular polygons. All we can deduce is that it will have the same number of edges as the coordination number P of the original vertex.

It is easy to see too that each face of the original solid will become a vertex and that the coordination number of each new vertex will be equal to the number of edges on the original polygon. There will therefore be only one type of face but two types of vertex. We need, therefore to change some definitions.

Notation:

N_A = number of vertices of type A on each face
N_B = number of vertices of type B on each face
P_A = number of faces meeting at each vertex of type A
P_B = number of faces meeting at each vertex of type B
V_A = number of vertices of type A
V_B = number of vertices of type B
F = number of faces
E = Number of edges

Theorem 1.3

$$\boxed{\begin{array}{c} V_A = 4P_BN_A/Z \quad : \quad V_B = 4P_AN_B/Z \\ Z = 2P_AP_B - P_AN_B(P_B-2) - P_BN_A(P_A-2) \end{array}}$$

Euler's theorem $\quad F+V_A+V_B-E = 2$

also $\quad F = (V_AP_A+V_BP_B)/(N_A+N_B)$

and $\quad E = F(N_A+N_B)/2 = (V_AP_A+V_BP_B)/2$

But $\quad (F =)\ V_AP_A/N_A = V_BP_B/N_B$

from which $\quad V_A = 4P_BN_A/Z \qquad V_B = 4P_AN_B/Z$

where $\quad Z = 2P_AP_B - P_AN_B(P_B-2) - P_BN_A(P_A-2)$

In general, we know the total number of vertices and their types already. What we don't know is the number of each type of vertex (ie N_A and N_B) and the shape of the faces. It is therefore easier to use the simpler relations:

$$N_A = N\frac{V_AP_A}{V_AP_A+V_BP_B} \qquad N_B = N\frac{V_BP_B}{V_AP_A+V_BP_B}$$

where $N = N_A + N_B$

The simplest Archimedean solid is the square drum which, you will remember, has 2 squares and 8 triangles – i.e. 10 faces and 8 vertices. Its complement will therefore have 10 vertices and 8 faces. In addition, we know that, since four faces surround each vertex in the drum, each face of the complimentary solid will have 4 vertices.

So we know that:

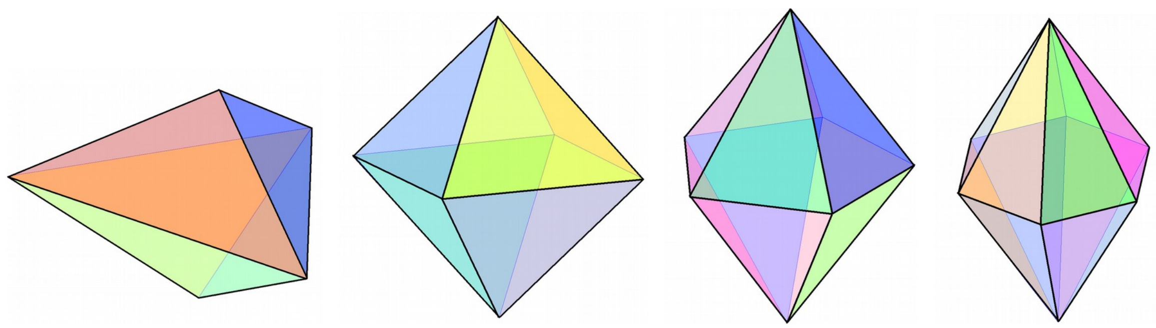
$N = N_A + N_B = 4$
$P_A = 3$ (the 'triangle' vertices)
$P_B = 4$ (the 'square' vertices)
$V_A = 8$
$V_B = 2$
$F = 8$

This gives $N_A = 3$ and $N_B = 1$

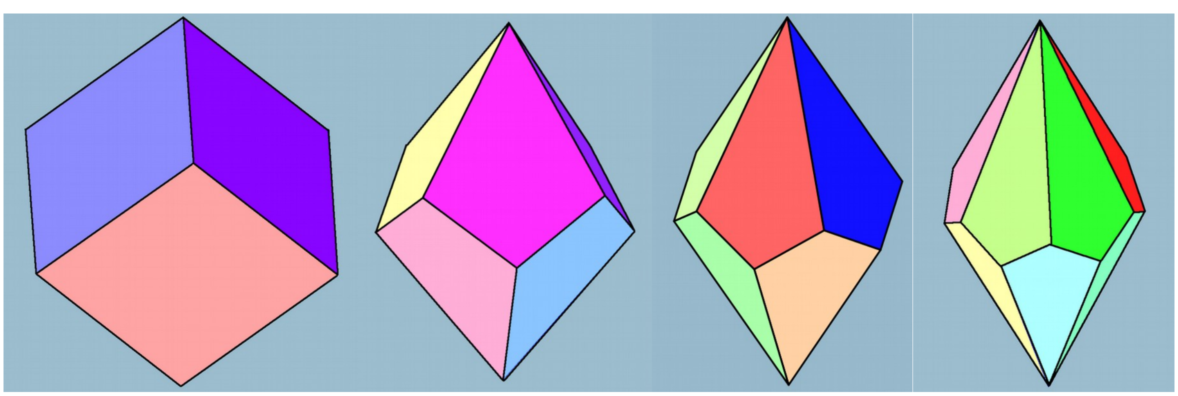
What we are looking for is a solid with 8 faces, 4 joined at an apex at the top, 4 at the bottom with 8 triple vertices round the waist. It is fairly clear that such a solid consisting of 8 quadrilaterals exists but our analysis does not tell us the precise shape of the faces, nor is it easy to see how they may be calculated. It is easy, however, to see that the faces must be deltoidal (kite-shaped).

In general, if you try sticking together 8 deltoids of arbitrary shape, it can be done but the deltoid will be bent in the middle. However, a deltoid has two degrees of freedom and if you fix one apex angle, there is always a second apex angle which will work.

The complements of the prisms turn out to be a series of bi-pyramids:



while the complements of the drums are deltohedra (ie solids whose faces are all deltoids.



The complements of the thirteen true Archimedean solids are called **Catalan** solids and can be seen on the following site:

http://dmccooey.com/polyhedra/index.html

Archimedean solids containing 3 types of polygon

Theorem 1.3

By analogy with the preceding formula and using the fact that it must reduce to that formula when $N_B = N_C$ or $P_C = 0$ we can suppose that

$$F_A = 4P_A N_B N_C / Z \qquad F_B = 4P_B N_A N_C / Z \qquad F_C = 4P_C N_A N_B / Z$$
$$Z = 2N_A N_B N_C - P_A N_B N_C (N_A - 2) - P_B N_A N_C (N_B - 2) - P_C N_A N_B (N_C - 2)$$

- *Consider those solids with triangles, squares and pentagons ie* $N_A = 3$, $N_B = 4$ *and* $N_C = 5$

The case 3.4.5.4 is consistent with the above formula and has 20 triangles, 30 squares, 12 pentagons and has 60 vertices. It is an enlarged dodecahedron with connecting rings of alternate triangles and squares.

Expanded Pentaball

- *Consider those solids with triangles, squares and hexagons ie* $N_A = 3$, $N_B = 4$ *and* $N_C = 6$

The cases 3.3.4.6 and 3.4.3.6 are consistent with the above formula and purport to have 16 triangles, 6 squares, 4 hexagons and has 24 vertices but the latter case is impossible to construct and the former turns out to be a truncated octahedron with three of the hexagons replaced by triangles.

- *Consider those solids with triangles, squares and heptagons ie* $N_A = 3$, $N_B = 4$ *and* $N_C = 7$

No combinations are consistent with the above formula.

- *Consider those solids with triangles, squares and larger polygons ie* $N_A = 3$, $N_B = 4$ *and* $N_C > 7$

The cases 3.3.4.8 and 3.4.3.8 are consistent with the above formula and purport to have 32 triangles, 12 squares and 6 octagons. The first case turns out to have flat hexagons (and is therefore the same as A4-6-8) while the second is impossible. In fact it is easy to see that all cases of the form 3-3-4-N end up with flat hexagons, and cases of the form 3-4-3-N are impossible to build.

There is one other possibility: 3-4-12 but this only has 1 large polygon and is impossible to build.

- *Consider those solids with squares, pentagons and another larger polygon ie* $N_A = 4$, $N_B = 5$ *and* $N_C > 5$

No combinations are consistent with the above formula except for $N_C = 10$ which has 10 squares, 8 pentagons and 4 decagons. This solid is not possible because it is not possible for squares and decagons to alternate round a pentagon.

- *Consider those solids with squares, hexagons and another larger polygon ie* $N_A = 4$, $N_B = 6$ *and* $N_C > 5$

4.6.7 is not consistent.

4.6.8 is possible and has 12 squares, 8 hexagons, 6 octagons and 48 vertices. It is a kind of Expanded truncated cube with bands of alternate squares and hexagons separating the octagonal sides.

Exploded Cube

4.6.9 is consistent with the formula but it is clear that it is impossible to build because the squares and hexagons must alternate round the large polygon which must therefore have an even number of sides.

4.6.10 has 30 squares, 20 hexagons, 12 decagons and 120 vertices. It is an Expanded version of a truncated dodecahedron consisting of bands of squares and hexagons separating 12 dodecahedra.

Exploded Pentaball

- *Consider those solids with squares and two larger polygons ie* $N_A = 4$, $N_B > 6$ *and* $N_C > 7$

It is easy to see that all the polygons must have an even number of sides. The smallest possibility is 4.8.10 which will not fit round a vertex

- *Consider those solids with even larger polygons ie* $N_A > 4$, $N_B > 6$ *and* $N_C > 7$

It is easy to see that all the polygons must have an even number of sides. The smallest possibility is 6.8.10 which will not fit round a vertex. There are no other possibilities with three polygons

N-polygon solids (N > 3)

The smallest case (3.4.5.6) will not fit round a vertex. There are therefore no more Archimedean solids.

Summary

Configur. ation	Type	F_A	F_B	F_C	V	Proper name	My name
3.3.3.4	Drum	8T	2S		8	Square Antiprism	Square Drum
3.3.3.3.4	Arch	32T	6S		24	Snub Cube	Dilated Cube
3.4.4	Prism	2T	3S		6	Triangular Prism	Triangular prism
3.4.3.4	Arch	8T	6S		12	Cuboctahedron	Chopped cube
3.4.4.4	Arch	8T	18S		24	Small Rhombicuboctahedron	Expanded Cube
3.3.3.5	Drum	10T	2P		10	Pentagonal Antiprism	Pentagonal Drum
3.3.3.3.5	Arch	80T	12P		60	Snub Dodecahedron	Dilated Pentaball
3.5.3.5	Arch	20T	12P		30	Icosidodecahedron	Chopped Pentaball
3.3.3.6	Drum	12T	2X		12	Hexagonal Antiprism	Hexagonal drum
3.6.6	Arch	4T	4X		12	Truncated Tetrahedron	Clipped Pyramid
3.3.3.7	Drum	14T	2H		14	Heptagonal Antiprism	Heptagonal drum
3.8.8	Arch	8T	6O		24	Truncated Cube	Clipped cube
3.10.10	Arch	20T	12D		60	Truncated Dodecahedron	Clipped Pentaball
4.4.5	Prism	5S	2P		10	Pentagonal Prism	Pentagonal Prism
4.4.6	Prism	6S	2X		12	Hexagonal Prism	Hexagonal Prism
4.6.6	Arch	6S	8X		24	Truncated Octahedron	Clipped Diamond
4.4.7	Prism	7S	2H		14	Heptagonal Prism	Heptagonal Prism
5.6.6	Arch	12P	20X		60	Truncated Icosahedron	Clipped Triball or Buckyball
4.6.8	Arch	12S	8X	6O	48	Large Rhombicuboctahedron	Exploded Cube
3.4.5.4	Arch	20T	30S	12P	60	Small Rhombicosidodecahedron	Expanded Pentaball
4.6.10	Arch	30S	20H	12D	120	Large Rhombicosidodecahedron	Exploded Pentaball

Note on my terminology

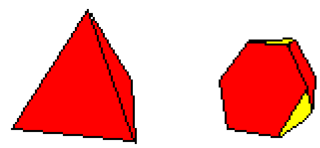
Clipped

This is synonymous with 'truncated' and means that all the vertices have been clipped off leaving all edges equal to approximately one third of the original length. It has the effect of increasing the number of faces by the number of original vertices, and multiplying the number of vertices by the number of faces round a vertex..

$$\begin{aligned} F' &= F+V \\ V' &= PV \\ N.N.N &\rightarrow P.2N.2N \end{aligned}$$

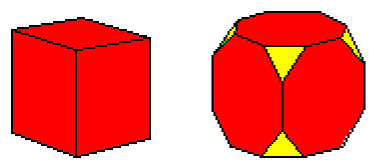
The Clipped Pyramid

$$\begin{aligned} F' &= 4+4=8 \\ V' &= 3\times4=12 \\ 3.3.3 &\rightarrow 3.6.6 \end{aligned}$$



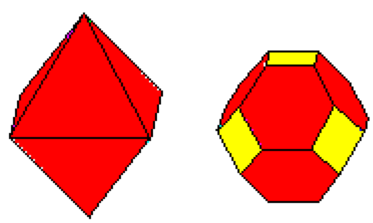
The Clipped Cube

$$\begin{aligned} F' &= 6+8=14 \\ V' &= 3\times8=24 \\ 4.4.4 &\rightarrow 3.8.8 \end{aligned}$$



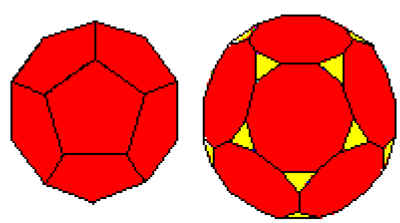
The Clipped Diamond:

$$\begin{aligned} F' &= 8+6=14 \\ V' &= 4\times6=24 \\ 3.3.3.3 &\rightarrow 4.6.6 \end{aligned}$$



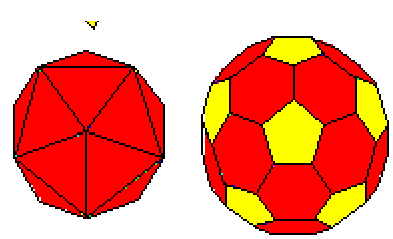
The Clipped Pentaball

$$\begin{aligned} F' &= 12+20=32 \\ V' &= 3\times20=60 \\ 5.5.5 &\rightarrow 3.10.10 \end{aligned}$$



The Clipped Triball

$$\begin{aligned} F' &= 20+12=32 \\ V' &= 5\times12=60 \\ 3.3.3.3.3 &\rightarrow 5.6.6 \end{aligned}$$



This is the Buckyball.

It is interesting to note that the clipped versions of complementary solids are themselves complementary in a different way; having the same number of faces and vertices, but halving and doubling the number of sides on the two kinds of polygon.

Chopped

This is a bit more drastic than 'clipping'; all the corners are chopped off completely leaving nothing left of the original edge. As with clipping, the number of faces increases by the number of original vertices but the number of vertices is increased by only half as much owing to the fact that every pair of new vertices are elided together.

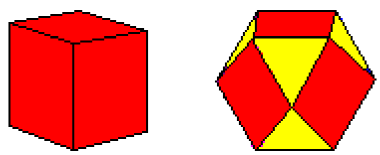
$$\begin{aligned} F' &= F+V \\ V' &= PV/2 \\ N.N.N... &\rightarrow P.N.P.N \end{aligned}$$

The Chopped Tetrahedron is an Octahedron

$$\begin{aligned} F' &= 4+4=8 \\ V' &= 3\times4/2=6 \\ 3.3.3 &\rightarrow 3.3.3.3 \end{aligned}$$

The Chopped Cube:

$$\begin{aligned} F' &= 6+8=14 \\ V' &= 3\times8/2=12 \\ 4.4.4 &\rightarrow 3.4.3.4 \end{aligned}$$



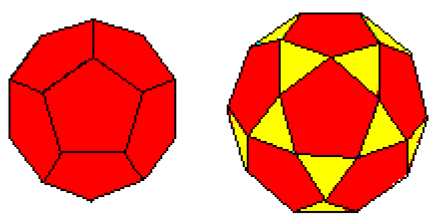
The Chopped Octahedron is also a Chopped Cube.

$$\begin{aligned} F' &= 8+6=14 \\ V' &= 4\times6/2=12 \\ 3.3.3.3 &\rightarrow 4.3.4.3 \end{aligned}$$

This is a consequence of the Cube and the Octahedron being complementary

The Chopped Pentaball (Chopped Dodecahedron)

$$\begin{aligned} F' &= 12+20=32 \\ V' &= 3\times20/2=30 \\ 5.5.5 &\rightarrow 3.5.3.5 \end{aligned}$$



The Chopped Icosahedron is also a Chopped Dodecahedron for the same reason

$$\begin{aligned} F' &= 20+12=32 \\ V' &= 5\times12/2=30 \\ 3.3.3.3.3 &\rightarrow 5.3.5.3 \end{aligned}$$

.

Dilated

This involves taking the original solid apart and joining them back together with chains of triangles. Each vertex becomes a new face (with $N = P$) (shown yellow below) and each edge becomes two new triangles (shown blue).

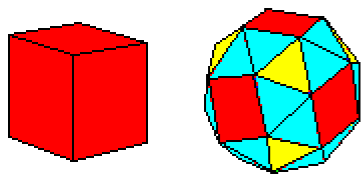
$$F' = F + V + 2 \times E$$
$$V' = N \times F$$
$$N.N.N... \rightarrow 3.3.P.3.N$$

The Dilated Tetrahedron is an Icosahedron.

$$F' = 4+4+2\times 6 = 20$$
$$V' = 3\times 4 = 12$$
$$3.3.3 \rightarrow 3.3.3.3.3$$

The Dilated Cube:

$$F' = 6+8+2\times 12 = 38$$
$$V' = 4\times 6 = 24$$
$$4.4.4 \rightarrow 3.3.3.3.4$$



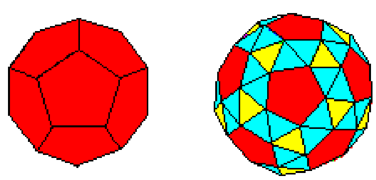
It is worth noting here that because of the 'skew' nature of this operation, the Dilated Cube has two chiral forms – ie a left and a right handed form.

The Dilated Octahedron is also a Dilated Cube.

$$F' = 8+6+2\times 12 = 38$$
$$V' = 3\times 8 = 24$$
$$3.3.3.3 \rightarrow 3.3.4.3.3$$

The Dilated Pentaball (Dilated Dodecahedron)

$$F' = 12+20+2\times 30 = 92$$
$$V' = 5\times 12 = 60$$
$$5,5,5 \rightarrow 3.3.3.3.5$$



The Dilated Icosahedron is also a Dilated Dodecahedron

$$F' = 20+12+2\times 30 = 92$$
$$V' = 3\times 20 = 60$$
$$3.3.3.3.3 \rightarrow 3.3.5.3.3$$

As before, complementary solids generate the same result when dilated.

Expanded

This is similar to dilation except that the edges are joined with a single square instead of two triangles.

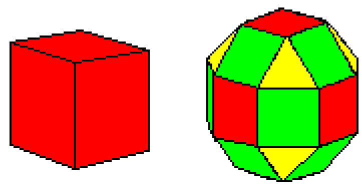
$$F' = F + V + E$$
$$V' = N \times F$$
$$N.N.N... \rightarrow N.4.P.4$$

The Expanded Tetrahedron is a Chopped Cube.

$$F' = 4+4+6 = 14$$
$$V' = 3 \times 4 = 12$$
$$3.3.3 \rightarrow 3.4.3.4$$

The Expanded Cube:

$$F' = 6+8+12 = 26$$
$$V' = 4 \times 6 = 24$$
$$4.4.4 \rightarrow 4.4.3.4$$

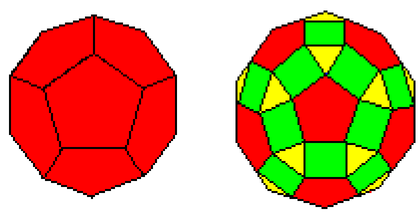


The Expanded Octahedron is also an Expanded Cube.

$$F' = 8+6+12 = 26$$
$$V' = 3 \times 8 = 24$$
$$3.3.3.3 \rightarrow 3.4.4.4$$

The Expanded Pentaball (Expanded Dodecahedron)

$$F' = 12+20+30 = 62$$
$$V' = 5 \times 12 = 60$$
$$5.5.5 \rightarrow 5.4.3.4$$



The Expanded Icosahedron is also an Expanded Dodecahedron.

$$F' = 20+12+30 = 62$$
$$V' = 3 \times 20 = 60$$
$$3.3.3.3.3 \rightarrow 3.4.5.4$$

Exploded

This can be thought of as being a two stage process: first the solid is clipped creating new faces at the corners; then the solid is expanded using new squares to join the edges.

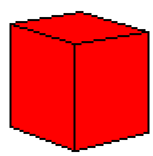
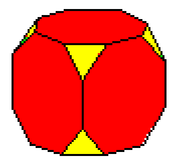
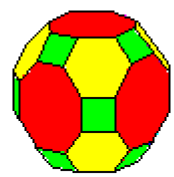
$$F' = F+V$$
$$V' = PV$$
$$F''=F'+E = F+V+E$$
$$V'' = 2\mathrm{V}' = 2\mathrm{PV}$$
$$N.N.N... \rightarrow 2\mathrm{N}.4.2\mathrm{P}$$

The Exploded Tetrahedron is a Clipped Octahedron

$$F'' = 4+4+6 = 14$$
$$V'' = 2\times3\times4 = 24$$
$$3.3.3 \rightarrow 6.4.6$$

The Exploded Cube:

$$F'' = 6+8+12 = 26$$
$$V'' = 2\times3\times8 = 48$$
$$4.4.4 \rightarrow 8.4.6$$

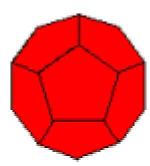
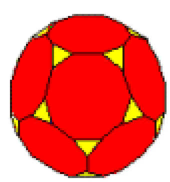
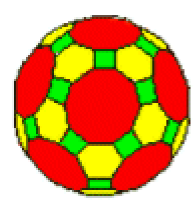




The Exploded Octahedron is (of course) the same as the Exploded Cube

$$F'' = 8+6+12 = 26$$
$$V'' = 2\times4\times6 = 48$$
$$3.3.3.3 \rightarrow 6.4.8$$

The Exploded Pentaball (Exploded Dodecahedron)

$$F'' = 12+20+30 = 62$$
$$V'' = 2\times3\times20 = 120$$
$$5.5.5 \rightarrow 10.4.6$$

The Exploded Icosahedron is (of course) the same as the Exploded Dodecahedron.

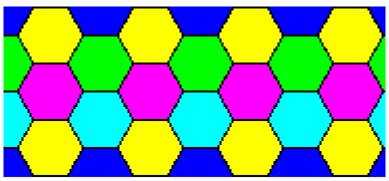
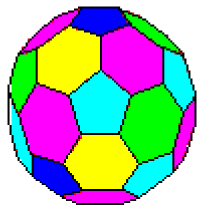
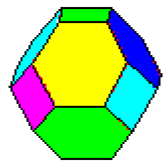
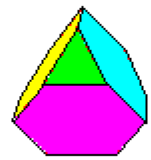
$$F'' = 20+12+30 = 62$$
$$V'' = 2\times5\times12 = 120$$
$$3.3.3.3.3 \rightarrow 6.4.10$$

N-polygon Tilings

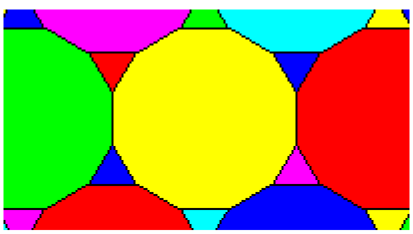
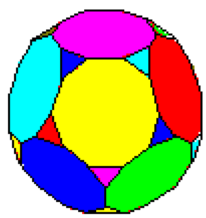
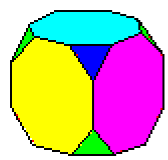
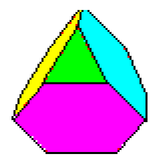
As has been noted above, a number of vertex configurations lead to plane tessellations namely (3.3.3.3.3.3), (3.3.3.4.4), (3.3.4.3.4), (3.3.3.3.6), (3.3.6.6), (3.6.3.6), (3.12.12), (3.4.3.6), (4.4.4.4), (4.8.8), (4.6.12). By successively reducing the number of sides of the largest polygon)s) you can often generate a series of semi-regular solids.

For example: starting with (6.6.6) we can of course generate the Pentaball (5.5.5), the Cube (4.4.4) and the Pyramid (3.3.3)

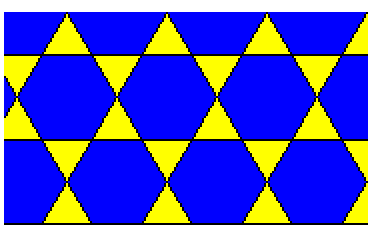
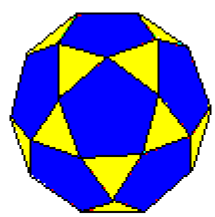
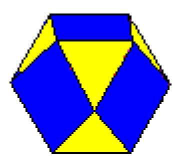
Alternatively, by reducing one polygon only we get first the Clipped Triball or Buckyball (5.6.6), the Clipped Diamond (4.6.6) and the Clipped Pyramid (3.6.6) whose general formula is (n.6.6).

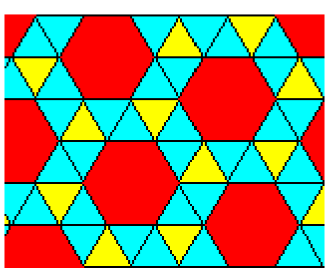
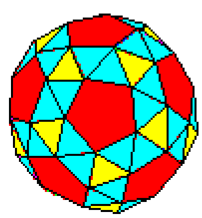
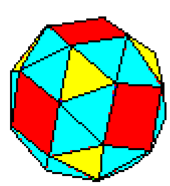
Starting with (3.12.12) you get the Clipped Pentaball (3.10.10), then the Clipped Cube (3.8.8), then the Clipped Pyramid (3.6.6). All these solids have the form (3.2n.2n).

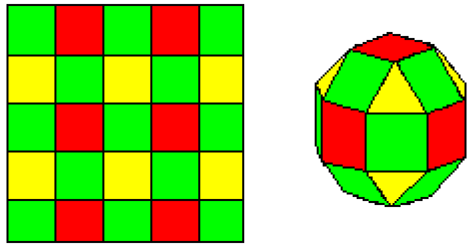
The tiling (3.6.3.6) turns into the Chopped Pentaball (3.5.3.5) then the Chopped Cube (3.4.3.4). The general formula is (3.n.3.n).

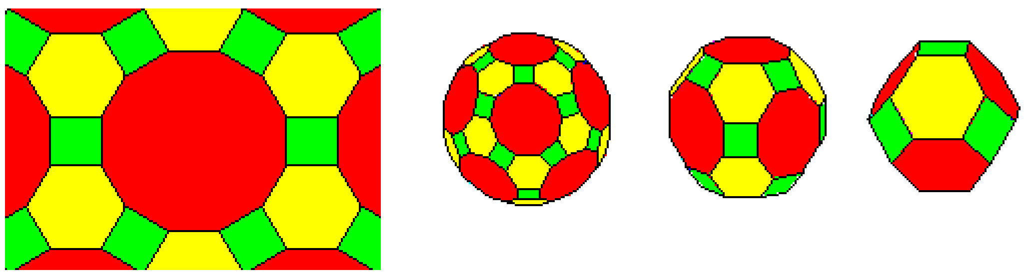
To generate the Dilated solids we need to start with the tessellation (3.3.3.3.6). This reduces to the Dilated Pentaball (3.3.3.3.5), then the Dilated Cube (3.3.3.3.4). The general formula is (3.3.3.3.n)

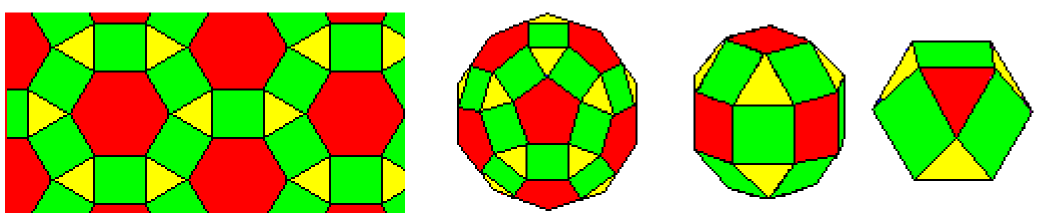
The tessellation 4.4.4.4 generates just one semi-regular solid – the Expanded Cube (4.4.4.3). The general formula for this series is (4.4.4.n).



To generate the 3.polygon solids we need to start with a 3.polygon tessellation. Here is the (4.6.12) tessellation. It reduces first to the Exploded Pentaball (4.6.10), then the Exploded Cube (4.6.8) and finally the Clipped Diamond (4.6.6). The formula for this series is (4.6.2n).



The last series begins with (3.4.6.4) and reduces to the Expanded Pentaball (3.4.5.4) before transforming into the Expanded Cube (3.4.4.4) and the Chopped Cube (3.4.3.4). The general formula is (3.4.n.4)



The general case

The formulae derived above may be generalized to any solid which has any number of different regular polygonal faces A, B, C.. [X]. and any number of different types of vertex 1, 2, 3... [k] as follows:

$$V_1 Z_1 + V_2 Z_2 + \ldots V_k Z_k \;=\; 4 N_A N_B \ldots N_X$$

where $$Z_k \;=\; 2 N_A N_B \ldots N_X \;-\; P_{kA} N_B N_C (N_A - 2) \;-\; P_{kB} N_A N_C (N_b - 2) \;-\; \ldots$$

The fact that all values of Z must be greater than zero puts severe constraints on the possible values of N and P.

Once the number of vertices has been decided upon, then number of faces can be worked out as follows:

$$F_X \;=\; (V_1 P_{1\mathrm{X}} \;+\; V_2 P_{2\mathrm{X}} \;+\; \ldots) / N_X$$

For example, let us try to find a solid made of triangles and squares ($N_A = 3$, $N_B = 4$) with two types of vertex $V_1 = (3.3.3)$ and $V_2 = (3.4.3.4)$. These parameters generate integral values of Z_1 (=12) and Z_2 (=4) and three possible pairs of values of V_1 and V_2, namely 1 and 9, 2 and 6, or 3 and 3 respectively. Of these, only the middle one generates integral numbers of faces and represents a solid having 6 triangles and 3 squares. This solid turns out to be a triangular prism whose ends have been capped with pyramids. I call these solids 'crystals' and this one is the triangular crystal. It is important to note that the existence of integral solutions to the above equations does not necessarily imply that the corresponding solid exists.

N_A	N_B	V_1	V_2	F_A	F_B	
3	-	3.3.3	3.3.3.3	6	-	Triangular diamond
3	-	3.3.3.3.3	3.3.3.3	10	-	Pentagonal diamond
3	4	3.3.3	3.3.4.4	6	3	Triangular crystal
3	4	3.3.3.3	3.3.4.4	8	4	Square crystal
3	4	3.3.3.3.3	3.3.4.4	10	5	Pentagonal crystal
3	4	3.3.3.4	3.3.4.4	8	3	