π and e

π and e are the two most important irrational constants in mathematics. π is well-known to every schoolchild as the ratio of the circumference to the diameter of a circle but e is far less well understood. Is there a simple way to explain e to the layperson?

e from compound interest

Suppose you invest £1000 at a rate of 5% per annum for 20 years. What return would you expect to net?

Your first thought might be to multiply 5% by 20 to get 100% and say that your investment would gain £1000 of interest and be worth £2000 at the end of the term.

But then you realize that the bank in giving you 5% *per annum* and that the interest gained every year is itself going to earn interest. If you take this into account, you will find that after the first year, your investment is worth £1050, after the second, it is worth £1050 × 1.05 = £1071, after the third: £124.55 etc. etc. Every year the investment increases by a *factor* of 1.05 so after n years it has increased by 1.05^n. We can easily calculate, therefore that after 20 years our investment will be worth £1000 × 1.05^{20} = £2653.29. Quite a significant advantage over our original estimate, don't you think?

It might occur to you now that if you could persuade the bank to do their interest calculation *every month* instead of every year, you could get even more money out of them. You try to persuade with the manager that 5/12% (=0.41666%) per month is the same as 5% per year. (Sadly, I think it very unlikely that he will agree with you but just let's suppose anyway!) Now your expected return is £1000 × 1.00416666^{240} = £2712.64 – Another £50 in your pocket!

Perhaps, by persuading the bank to calculate the interest *every day* or even *every minute* you could get even more money out of them! Lets see.

We started with the idea that if you invest a sum of money at 5% for 20 years you would earn 100% interest. Lets generalise this idea to investing at $100/n$ % for n intervals of time (years, months, day, whatever). Our investment will grow by a factor of $(1 + 1/n)$ every interval (in our first calculation, n was 20 years so the annual factor was $(1 + 1/20) = 1.05$) and after n intervals it will have multiplied by a factor of $(1 + 1/n)^n$.

Let us calculate this factor for some values of n

1	20	240	1000	1000000
2.00000	2.6533	2.71264	2.71692	2.71828

We see a remarkable thing. as we increase n, the interest factor increases – but it does so more and more slowly, levelling off at a figure of about 2.71828.

What this means is that, even if you bank compounded the interest *every second* or even every *microsecond*, the maximum interest it would pay would be 2.71828 times the capital (in the same period of time that the investment would double under simple interest.)

This magic number is e.

It is defined as the limit of $(1 + 1/n)^n$ as n tends to infinity. i.e.:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \tag{1}$$

The mathematics of growth

e is fundamental to the mathematics of any sort of growth where the *percentage* increase is constant.

Suppose a population increases by *p*% per year. Suppose that the population starts at y_0 at year 0. What will the population be in year *x*? The answer is, of course,

$$y = y_0 \left(1 + \frac{p}{100}\right)^x$$

Let's write 1 + p/100 as a simple factor *f*. Then we have

$$y = y_0 f^x$$

This kind of growth is called *exponential* growth because *x* appears as the *exponent* in the formula.

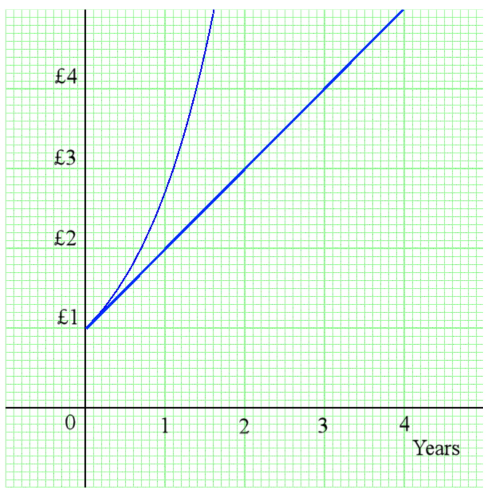
The remarkable properties of the function e^x

One day in the newspaper you come across the following advertisement:

> Once in a lifetime offer!
> The **Royal Bank of Skittleland** is offering you
> ***100% per annum interest on £1****
> and as a special inducement, if you reply within 24 hours,
> we will ***compound*** the interest!
>
> * Participators in the scheme must also invest £1,000,000 at the derisory interest rate of 0.1% for the same term
> Maximum length of term: 10 years

We now know what this fantastic offer means. At the end of 1 year, if the interest was not compounded, we would earn 100% and our investment would be worth just £2 – and after 10 years it would be worth £11. But if the interest is compounded, after 1 year it will be worth £2.718 or £*e*; after 2 years it will be worth £e^2; after 3 years £e^3 etc. etc. After *x* years it will be worth £e^x. After 10 years, your £1 will be worth a very respectable £22,026. (If you are tempted by this offer, it is always worth looking at the small print!)

Lets compare things using a graph.



The simple interest curve grows at a steady rate if £1 per year but the compound interest rate increases by a *factor* of 2.718 *times* every year. That is the crucial difference. But I want to draw you attention to a singular fact. *Both curves start off in exactly the same way*. In the first few days, they grow at the same rate.

Mathematically speaking, the significance of this is profound. What we are saying is that *the gradient of the* e^x *curve at* $x = 0$ *is* 1. Now that may not sound very profound but it becomes more significant when we realise that *the gradient at* $x = 1$ *must be e* – and *the gradient at* $x = 2$ *must be* e^2 etc. etc.. In fact we have stumbled on the most remarkable property of *e* of all:

> The gradient of the curve $y = e^x$ at every point is equal to e^x.

Indeed, this may be taken to be a definition of *e* and we can even use it to calculate the value of *e*.

e as the sum of a series

You may remember from your school days that in order to calculate the gradient of a function like $y = 3x^2 + 4x - 1$ you have to perform some magic called *differentiation*. This involves multiplying the coefficient of each term by the exponent and then reducing the exponent by 1. In the case above the result is $\dot{y} = 6x + 4$. Lets not worry too much about the details, the long and short of it is that the gradient of a quadratic equation is a linear one; the gradient of a quartic is a cubic etc .etc. It is obvious therefore that no finite polynomial can be the same as its gradient because it is always one temr short.

But what about an infinite polynomial? Can we find a polynomial expression such that it equals its own gradient? If we reduce every exponent by 1 we are still left with an infinite series. What a bizarre idea! But it works! Consider:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

First we will make it go through the point (0,1) like the exponential graph by putting $a_0 = 1$

Now lets differentiate the whole thing.

$$\dot{f}(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$$

If this expression is going to be the same as the previous one, we must have the following identities:

$$\begin{aligned} a_1 &= a_0 \\ 2a_2 &= a_1 \\ 3a_3 &= a_2 \\ 4a_4 &= a_3 \\ &\text{etc.} \end{aligned}$$

from which it is easy to see that

$$\begin{aligned} a_1 &= 1 \\ a_2 &= \frac{1}{2} \\ a_3 &= \frac{1}{2}\cdot\frac{1}{3} \\ a_4 &= \frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2} \\ &\text{etc.} \end{aligned}$$

and that

$$a_n = \frac{1}{n!}$$

We now know of two functions which pass through the point (0,1) and which equal their own gradient. These functions *must* be the same so we conclude that:

$$e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

(I have written the first term a 1/0! to emphasise the symmetry of all the terms. To make this work, we must assume that factorial 0 is equal to 1)

Let's check this out with a few known values. Obviously when $x = 0$, the function reduces to 1 so that's OK.

When $x = 1$, we have

$$f(1) = \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+ \ldots = 1+1+0.5+0.166+0.042 \ \ldots \approx 2.708$$

which seems to be converging very nicely on e.

When $x = 2$, we have

$$f(2) = \frac{1}{0!}+\frac{2}{1!}+\frac{4}{2!}+\frac{8}{3!}+\frac{16}{4!}+ \ldots = 1+2+2+1.33+0.666 \ \ldots \approx 7.000$$

which is approaching e^2.

So we now have our second method of calculating e. It is the sum of the following infinite series:

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \qquad (2)$$

Can we show that these two definitions are equivalent? Yes we can.

Consider the sequence

$$s_n = \left(1 + \frac{1}{n}\right)^n$$

We can expand the bracket using the binomial theorem:

$$s_n = 1 + \frac{n}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \ldots$$

$$s_n = 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + \ldots$$

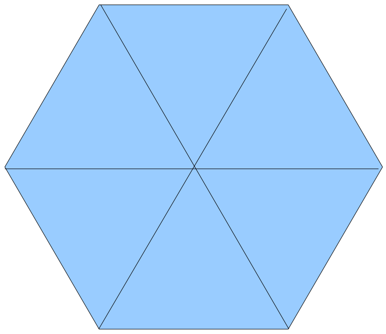
Now as n tends to ∞, all the bracketed terms tend to 1 – hence

$$s_\infty = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \ldots = e$$

This establishes the identity of equations (1) and (2)

π as a limit

Is there any way we can calculate π as a limit of a simple expression like equation (2). Unfortunately, the answer is no. Archimedes calculated π quite accurately centuries ago by calculating the perimeter of a polygon with 96 sides and it is easy to see that the more sides a polygon has, the more closely will the ratio of its perimeter to its diameter approach π. The trouble is, there is no simple formula for the length of the side of an n-sided polygon. We can, however, make a start with a 6-sided polygon:



The perimeter is obviously 6 units and the diameter 2 so our first approximation to π is 3.

Now suppose, we knock each side out a bit to make it into a 12-sided polygon made of 12 isosceles triangles whose apex angle is 30^0. The perimeter will now be

$$12 \times (2\sin 15^0) = 6.2116$$

and our next approximation to π is 3.1058. (The base of an isosceles triangle of unit side and apex angle θ is 2 sin $\theta/2$)

The problem with this is that the calculator we used to calculate the sin of the angle almost certainly used a value of π to get the answer so our reasoning is circular and therefore invalid. (To be honest, when we used a calculator to calculate 1.05^{20} it may well have used a value of e so that was circular as well! On the other had, you *could* have done the calculations by long hand, couldn't you. *Couldn't you*? Oh well. It could be done.)

What we need is a way of calculating sin 15^0 without using a calculator. Now it is fairly easy to prove the following relation between the sin of an angle and the sine of half that angle:

$$\sin(\theta/2) = \sqrt{½(1 - \sqrt{1 - \sin^2\theta})}$$

We know (by Pythagoras' theorem) that sin 30^0 = 0.5 so we can easily calculate sin 15^0 and it comes to 0.2588 giving the value for π quoted above. And with the value of sin 15^0, we can calculate the value of sin 7.5^0 and hence the perimeter of a 48-sided polygon etc. etc. each time getting a better and better value for π.

In practice, this is a terrible way of calculating π but at least you can see how, in principle, it can be done.

A series for π

Let us try to find an infinite series for the functions sin(x) and cos(x). What do we know about them?

Just as we were able to define the function e^x as that function which equals its own gradient, we can define sin(x) and cos(x) as that pair of functions which have the following relations:

The gradient of sin(x) is cos(x)
The gradient of cos(x) is -sin(x)
The gradient of -sin(x) is -cos(x)
The gradient of -cos(x) is sin(x)

We also know that sin(0)=0 and cos(0)=1 so we can start by assuming that

$$\sin(x) \;=\; 0 \;+\; a_1 x \;+\; a_2 x^2 \;+\; a_3 x^3 \;+\; a_4 x^4 \;+\; \ldots$$
$$\cos(x) \;=\; 1 \;+\; b_1 x \;+\; b_2 x^2 \;+\; b_3 x^3 \;+\; b_4 x^4 \;+\; \ldots$$

First we note that the sin function returns to minus itself after *two* diferentiations and itself after *four* diferentiations. Since the first term is zero, *all* the even coefficients must be zero as well.

Now since sin turns into cos after one differentiation, all the *odd* coefficients of cos(x) must be zero too.

Adding in the fact that two differentiations turns sin into minus sin, we have to conclude that the signs alternate like this:

$$\sin(x) \;= a_1 x \;-\; a_3 x^3 \;+\; a_5 x^5 \;-\; a_7 x^7 \;+\; \ldots$$
$$\cos(x) \;=\; 1 \;-\; b_2 x^2 \;+\; b_4 x^4 \;-\; b_6 x^6 \quad \ldots$$

Further investigation reveals that

$$\sin(x) \;= x \;-\; \frac{1}{3!}x^3 \;+\; \frac{1}{5!}x^5 \;-\; \frac{1}{7!}x^7 \;+\; \ldots$$
$$\cos(x) \;=\; 1 \;-\; \frac{1}{2!}x^2 \;+\; \frac{1}{4!}x^4 \;-\; \frac{1}{6!}x^6 \;+\; \ldots$$

Unfortunately, while these formulae are of great importance, they do not really help us to work out a value for π . While we know that sin($\pi/6$) = 0.5, we can't solve an infinite polynomial to work out its solution (though we can use numeric methods to home in on the answer). Putting $x = \pi/6$ into the formula for sin(x) we get:

$$\sin\left(\frac{\pi}{6}\right) = 0.523 \;-\; \frac{1}{3!}0.523^3 \;+\; \frac{1}{5!}0.523^5 \;-\; \frac{1}{7!}0.523\mathrm{x}^7 \;+\; \ldots \;\approx\; 0.500$$

so it works all right.

On the other hand, a formula for arctan(x) (the angle whose tangent is x) is just what we need. This formula is:

$$\arctan(x) \;=\; x \;-\; \frac{x^3}{3} \;+\; \frac{x^5}{5} \;-\; \frac{x^7}{7} \;-\; \ldots$$

Since the tangent of $\pi/4$ is 1, we can say that

$$\pi \;=\; 4 \;\times\; \left(1 \;-\; \frac{1}{3} \;+\; \frac{1}{5} \;-\; \frac{1}{7} \;-\; \ldots\right) \;\approx\; 2.895$$

The only trouble is – we need to take about 100 terms of this series to calculate even a couple of decimal places. Not to worry, though. There are much better series formulae which converge on the answer much more quickly and π has been calculated to over a trillion digits.

The connection between e and π

We have seen how both the exponential functions and the trigonometrical functions can be expressed as an infinite polynomial on the basis of the way they behave when they are differentiated. Let us put the three polynomials side by side.

$$\begin{array}{lllllllllll} e^x & = & 1 & + ix & + \frac{1}{2!}x^2 & + \frac{1}{3!}x^3 & + \frac{1}{4!}x^4 & + \frac{1}{5!}x^5 & + \dots \\ \sin(x) & = & & x & & - \frac{1}{3!}x^3 & & + \frac{1}{5!}x^5 & - \dots \\ \cos(x) & = & 1 & & - \frac{1}{2!}x^2 & & + \frac{1}{4!}x^4 & & - \dots \end{array}$$

All the same terms are there but, because of these minus signs, we cannot just add sin(x) and cos(x) to get e^x.

The trick is to calculate e^{ix} not e^x. i (the square root of -1) behaves exactly like the sine and cosine functions. You need to multiply itself *four* times to get back to where you started from.

$$\begin{array}{lllllllllll} e^{ix} & = & 1 & + ix & - \frac{1}{2!}x^2 & - \frac{i}{3!}x^3 & + \frac{1}{4!}x^4 & + \frac{i}{5!}x^5 & + \dots \\ \sin(x) & = & & x & & - \frac{1}{3!}x^3 & & + \frac{1}{5!}x^5 & - \dots \\ \cos(x) & = & 1 & & - \frac{1}{2!}x^2 & & + \frac{1}{4!}x^4 & & - \dots \end{array}$$

Now the minus signs line up exactly where we need them. All we have to do is throw in an extra i and we arrive at probably the most remarkable formula in all of mathematics.

$$\boxed{e^{ix} = \cos(x) + i\sin(x)}$$

and putting $x = \pi$

$$\boxed{e^{i\pi} = -1}$$