Permutations and Combinations

Permutations

1) How many ways are there of putting n different items in order?

There are n ways of choosing the first item, n - 1 ways of choosing the second, n - 2 ways of choosing the third etc. etc. The total number of possible ways is therefore $n \times (n-2) \times (n-3) \times \ldots$ or:

$$O(n) = n!$$

eg there are 10! = 3,628,800 ways of putting the digits 0-9 in order. Incidentally, this means that out of the 10,000,000,000 10-digit numbers, only 0.03628% have no repeating digits.

2) How many ways are there of putting n items of which k are identical and n – k are different in order?

Since there are $k!$ ways of ordering the k identical items, the answer is $n! / k!$

eg there are 10! / 4! = 151,200 ways of putting the letters ABCDEFGGGG into order.

3) How many different ways are there of choosing an ordered list of k items from a pool of n different items?

There are n ways of choosing the first item, n - 1 ways of choosing the second, n - 2 ways of choosing the third etc. etc. The total number of possible ways is therefore $n \times (n-2) \times (n-3) \times \ldots \times (n-k+1)$. This is known as a *permutation* of k items from n and can be written as:

$$P(n,k) = \frac{n!}{(n-k)!}$$

eg there are 10! / 6! = 5,040 ways in which the first 4 places can be filled from a race with 10 starters.

Theorem 1: The number of ways in which n items (of which k are identical) can be put in order is equal to the number of ways of selecting an ordered list of n - k items from a pool of n different items - $P(n, n - k)$

eg suppose $n = 5$ and $k = 3$. There are 20 ways of putting the letters ABCCC in order and 20 ways of selecting ordered pairs of letters from the pool ABCDE. Here is a list

AB	ABCCC
AC	ACBCC
AD	ACCBC
AE	ACCCB
BA	BACCC
BC	BCACC
BD	BCCAC
BE	BCCCA
CA	CABCC
CB	CACBC
CD	CACCB

CE	CBACC
DA	CBCAC
DB	CBCCA
DC	CCABC
DE	CCACB
EA	CCBAC
EB	CCBCA
EC	CCCAB
ED	CCCBA

4) How many ways are there of putting k different balls into n different boxes (each of which can hold only one ball)?

There are *n* ways of placing the first item in a box, *n* - 1 ways of placing the second, *n* - 2 ways of placing the third etc. etc. The total number of possible ways is therefore *P*(*n*, *k*).

Theorem 2: The number of ways in which *k* different balls can be put into *n* different boxes is the same as the number of ways in which an ordered list of *k* items can be selected from a pool of *n* different items - *P*(*n*, *k*).

5) How many ways are there of selecting an unordered heap of k items from a pool of n different items?

Since there are *P*(*n*, *k*) ways of selecting ordered heaps, there will be *P*(*n*, *k*) / *n!* ways of selecting unordered heaps. This is known as a *combination* of *k* items from *n* and can be written as

$$C(n,k) = \frac{n!}{k!(n-k)!}$$

eg there are 10! / (4! × 6!) = 30,240 ways of dividing a pool of 10 items into a heap of 4 (and a heap of 6) items. It is immediately apparent that this function is symmetrical with respect to *k* and *n* – *k*.

6) How many ways are there of selecting two unordered heaps of j and k items from a pool of n different items?

We start by selecting *j* items from *n*. This gives us *C*(*n*, *j*) ways. Then we select *k* items from *n* – *j* ie *C*(*n* – *j*, *k*) ways. The total number is the product of these two ie:

$$C(n,j,k) = \frac{n!}{j!(n-j)!} \times \frac{n-j!}{k!(n-j-k)!} = \frac{n!}{j!k!(n-j-k)!}$$

Clearly this function can be generalised to give the number of ways in which any number of heaps may be selected from a pool of *n* items.

It is also the answer to the related question of how many ways can two heaps of *j* and *k* items be placed in *n* different boxes.

7) How many ways are there of putting k identical balls into n different boxes (each of which can contain only one ball?

The placing of a ball in one of the boxes can be regarded as the selection of that box and since the balls are identical, the order of the selection is irrelevant. We therefore conclude that

Theorem 3: The number of ways in which *k* identical balls can be put into *n* different boxes is the same as the number of ways in which a heap of *k* items can be selected from a pool of *n* different items - *C*(*n*, *k*).

8) How many ways are there of putting k identical balls into n identical boxes (each of which can contain only one ball?

The answer to this is trivial as there is only one way of doing it.

9) How many ways are there of putting k identical balls into n identical bags (each of which can contain any number of balls)?

There are lots of ways of doing this. You could put all the balls into one bag; you could put all but one ball into one bag and the remaining ball in another; you could put all but two in one bag and either put the remaining two into another bag or you could separate them etc. etc.

It is worth noting that if $n > k$, the number of ways does not increase with increasing n so the answer is not a simple formula valid for all n and k. Here is a list of a few small cases:

k	*n*	*N*	
1	1	1	1
2	1	1	1
2	2	2	20; 11
3	1	1	3
3	2	2	30; 21
3	3	3	300; 210; 111
4	1	1	4
4	2	3	40; 31; 22
4	3	4	400; 310; 220; 211
4	4	5	4000; 3100; 2200; 2110; 1111
5	1	1	5
5	2	3	50; 41; 32
5	3	4	500; 410; 320; 311; 221
5	4	6	5000; 4100; 3200; 3110; 2210; 2111
5	5	7	50000; 41000; 32000; 31100; 22100; 21110;11111

The pattern is not at all obvious and boils down to counting the number of ways in which a number k can be broken down into uniquely different addition sums.

10) How many ways are there of putting k different balls into n different bags? (A bag can contain more than one ball)

Since the bags can contain any number of balls, the answer is clearly n^k.

11) How many ways are there of putting k identical balls into n different bags?

This is a very different question and is a lot more difficult to answer. Since some of the combinations will have multiple balls in the same bag, these will be counted more than once. But we cannot simply divide by a factorial because some combinations will have pairs of balls, some triplets, others multiple pairs etc etc. The total number of combinations will therefore be a good deal less than n^k.

On the other hand, if we consider a similar but much easier question – how many ways are there of putting k identical balls into n boxes – each of which can contain only one ball (whose answer is obviously the same as selecting k items from a pool of n – ie $C(n,k)$) it is clear that there must be more ways than this because we can put more than one ball into each bag.

Let us suppose there are 10 bags (numbered 0 - 9) and 4 balls. Let us also suppose that we have a supply of extra bags (labelled A, B, C etc.).

Now take any valid combination of 4 balls in 10 bags. There are 8 possible cases which we must consider

- Four bags are used and there are no multiple bags 000
- Three bags are used and the lowest numbered bag contains a pair 100
- Three bags are used and the middle numbered bag contains a pair 010
- Three bags are used and the highest numbered bag contains a pair 001
- Two bags are used and the lowest numbered bag contains a triplet 110
- Two bags are used and the highest numbered bag contains a triplet 011
- Two bags are used and both bags contain a pair 101
- One bag is used and it contains all four balls 111

On the right I have coded each of these 8 possible cases with a binary number which suggests the case which it represents. There is 1 way of using one bag, 3 ways of using two bags, 3 of using three bags and 1 way of using four bags. Note that these are binomial coefficients.

With 5 balls instead of 4, there are 16 cases. grouped, as you would expect in sets of 1, 4, 6, 4 and 1.

Let us accept here without proof that there are always 2p multiple cases where p is one less than the number of balls and that there is always a unique way of coding these 2p cases with a p digit binary number.

Now let us go back to our valid combination of 4 balls in 10 bags. Pick up 3 extra bags A, B and C and place them on the end of the line. Fill those bags with the extra balls which occupy the multiple bags according to the code given above. (It will be noted that in the three cases where three bags are used and where there is just one extra ball, the binary number contains just one digit. Likewise in the two bag cases the number has 2 digits and in the last case it has 3. Again we shall assume without proof that the coding always works so that the binary number always contains exactly the right number of digits to accommodate all the extra balls!)

We have already established (admittedly without proof) that there is a one-to-one correspondence between the 4 balls in 10 (possibly multiple) bags case and the 4 balls in the 13 (singly occupied) bags case so the solution to our problem is simply C (13,4) which works out to be 13×12×11×10 / 4×3×2×1 = 715

In general, the number of ways of placing *k* identical balls into *n* different bags is

$$C(n+k-1,k) = \frac{n+k-1!}{k!(n-1)!}$$

This suggests the following theorem:

Theorem 4: The number of ways in which *k* identical items can be put into *n* different bags is equal to the number of ways in which *k* identical items can be placed into *n* + *k* – 1 boxes.

Summary

Case 1: identical balls into identical boxes

There is only one way to put 3 identical balls into 5 identical boxes and only one way of putting *k* items into *n* identical boxes

Case 2: identical balls into different boxes

There are 10 ways of putting 3 identical balls into 5 different boxes because this is equivalent to selecting a heap of three items out of a list of 5. In general, therefore, there are $C(n, k)$ ways of putting *k* identical items into *n* different boxes.

Case 3: different balls into identical boxes

There is only one way to put 3 identical balls into 5 identical boxes and only one way of putting *k* items into *n* identical boxes

Case 4: different balls into different boxes

There are 60 ways of putting 3 different balls into 5 different boxes. Ball number 1 can go in any one of 5 boxes; ball number 2 into one of 4; ball number 3 into one of 3 and $5 \times 4 \times 3 = 60$. In general, there are $P(n, k)$ ways of putting *k* different items into *n* different boxes.

Case 5: identical balls into identical bags

There are just three ways of putting 3 identical balls into 5 identical bags but the general rule is not at all obvious and amounts to counting up different ways of doing addition sums.

Case 6: identical balls into different bags

There are 35 ways of putting 3 identical balls into 5 different bags. This is because there are 5 ways of putting all the balls into the same bag; 20 ways of dividing the balls 2-1 and 10 ways of dividing them 1-1-1.

In general, for each way of uniquely dividing the k balls into different heaps, there will be a different number of ways in which these heaps can be placed in the bags and all these ways must be added together but because of theorem 4, this is equal to $C(n + k - 1, k)$

Case 7: different balls into identical bags

There are just five ways of putting 3 different balls into 5 identical bags. As with case 5 you must first start by figuring out all the different ways in which up to 5 numbers can sum to 3; then some of these can be permuted because the balls are different. For example, there are 3 different ways in which the balls can be split into a 2 and a 1.

Case 8: different balls into different bags

There are $5^3 = 225$ ways of putting 3 different balls into 5 different bags. In general, there are n^k ways of putting k different items into n different bags.