Kappa functions

The exponential function

The series expansion of e^x is:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \ldots \tag{1}$$

and it has the remarkable (and, apart from a constant of integration, unique) property that

$$\frac{d(e^x)}{dx} = e^x \tag{2}$$

In other words, the exponential function is the only function which returns to itself when differentiated once. It is therefore the solution to the equation:

$$\frac{dy}{dx} = y$$

The hyperbolic functions

Cosh(x) and sinh(x) have the following expansions:

$$\cosh(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \ldots$$
$$\sinh(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots$$

from which it is easy to see that

$$\frac{d(\cosh(x))}{dx} = \sinh(x)$$
$$\frac{d(\sinh(x))}{dx} = \cosh(x)$$

This means that the sinh and cosh functions return to themselves when differentiated twice and are the solutions to the equation:

$$\frac{d^2y}{dx^2} = y$$

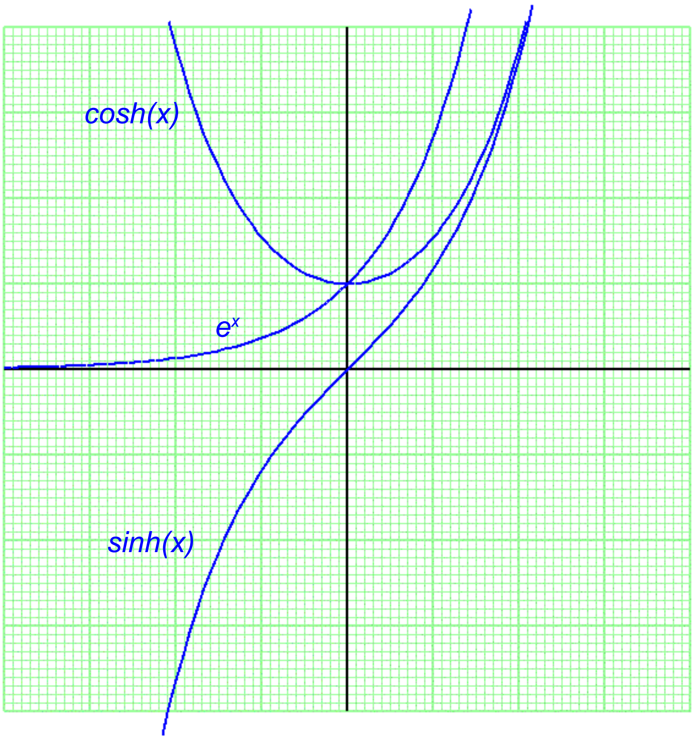
It is also easy to see that:

$$\cosh(x) + \sinh(x) = e^x$$

and that

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

The relationships can best be visualised on a graph:



Note how each graph is the gradient of the other and that, added together, they make e^x.

The question then suggests itself – what, in general, are the solutions to the equation

$$\frac{d^n y}{dx^n} = y$$

and what do they look like. Also – where do the trig functions sin and cos appear in this?

The trig functions

Cos x and sin x have the following expansions:

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots$$
$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots$$

The alternating minus signs add a new complication

$$\frac{d(\cos(x))}{dx} = -\sin(x)$$
$$\frac{d(\sin(x))}{dx} = -\cos(x)$$

This means that the sin and cos functions return to the negative of themselves when differentiated twice and are the solutions to the equation:

$$\frac{d^2 y}{dx^2} = -y$$

They are also solutions to the equation:

$$\frac{d^4 y}{dx^4} = y$$

but they are not unique in this respect.

They can be written as:

$$\cos(x) = \frac{e^{ix} + e^{-tx}}{2}$$
$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2\mathrm{i}}$$

The Kappa₃ function

Consider the three functions:

$$k_0(x) = 1 + \frac{1}{3!}x^3 + \frac{1}{6!}x^6 + \frac{1}{9!}x^9 + \ldots$$
$$k_1(x) = x + \frac{1}{4!}x^4 + \frac{1}{7!}x^7 + \frac{1}{10!}x^{10} + \ldots$$
$$k_2(x) = \frac{1}{2}x^2 + \frac{1}{5!}x^5 + \frac{1}{8!}x^8 + \frac{1}{11!}x^{11} + \ldots$$

It is easy to see that each function returns to itself after differentiation three times.

By analogy with the expressions for other functions, we can see that

$$k_0(x) = \frac{e^x + e^{\kappa_3 x} + e^{\kappa_3^2 x}}{3}$$
$$k_1(x) = \frac{e^x + \kappa_3 e^{\kappa_3 x} + \kappa_2^2 e^{\kappa_3^2 x}}{3}$$
$$k_2(x) = \frac{e^x + \kappa_3^2 e^{\kappa_3 x} + \kappa_3^4 e^{\kappa_3^2 x}}{3}$$

where $\kappa_3 = \sqrt[3]{1} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

(I have called functions like this 'kappa' functions because they all depend in a crucial way on the primary n$^{\text{th}}$ root of 1 – which I like to call κ_n)

After the next differentiation, the $\kappa_3^3 \text{and} \kappa_3^6$ coefficients disappear, returning us to $k_0(x)$ again.

Now

$$e^{\kappa_3 x} = e^{(-1/2 + \sqrt{3}/2\ i)}x = e^{-1/2\,x}e^{\sqrt{3}/2\ ix} = e^{-1/2\,x}\left(\cos(\alpha x) + i\sin(\alpha x)\right)$$

where $\alpha = \sqrt{3}/2$

Since κ_3^2 is the complex conjugate of κ_3 the imaginary parts of the expression will cancel out and we can write:

$$K_3(x) = \frac{e^x + e^{\kappa_3 x} + e^{\kappa_3^2 x}}{3} = \frac{e^x + 2e^{-1/2\,x}\cos(\alpha x)}{3}$$

It is of interest to check this expression by differentiating it three times. Ignoring the e^x term and the numerical constants, we get first:

$$-1/2\,e^{-1/2\,x}\cos(\alpha x) - \alpha e^{-1/2\,x}\sin(\alpha x)$$

then:

$$1/4\ e^{-1/2\,x}\cos(\alpha x) + \alpha/2\ e^{-1/2\,x}\sin(\alpha x) + \alpha/2\ e^{-1/2\,x}\sin(\alpha x) - \alpha^2\ e^{-1/2\,x}\cos(\alpha x)$$
$$(1/4 - \alpha^2)\ e^{-1/2\,x}\cos(\alpha x) + \alpha\ e^{-1/2\,x}\sin(\alpha x)$$

and finally:

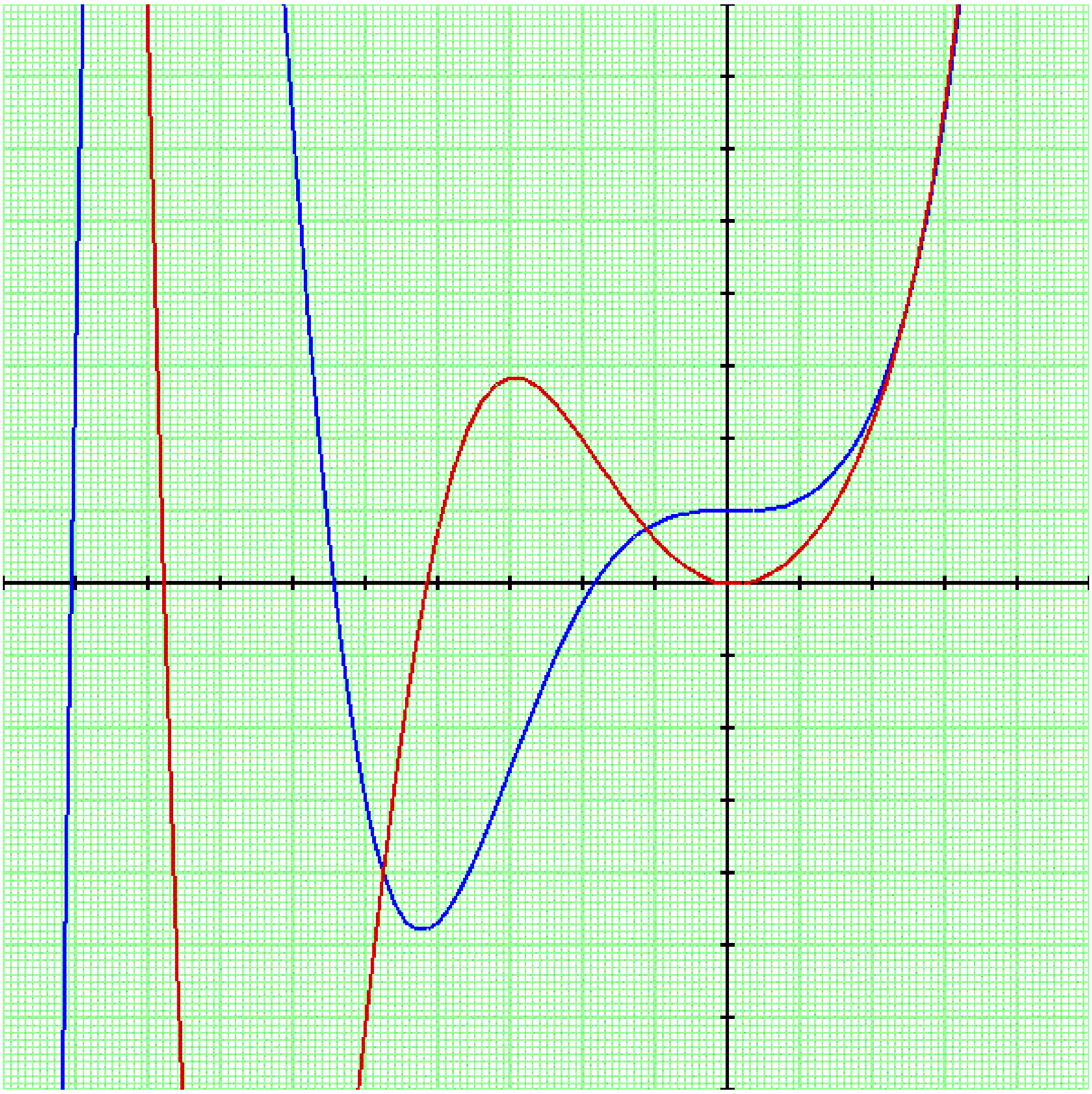
$$(\alpha^2-1/4)/2\,e^{-1/2\,x}\cos(\alpha x)-\alpha(1/4-\alpha^2)e^{-1/2\,x}\sin(\alpha x)-\alpha/2\;\;e^{-1/2\,x}\sin(\alpha x)+\alpha^2\;\;e^{-1/2\,x}\cos(\alpha x)$$

$$(3\alpha^2-1/4)/2\;\;e^{-1/2\,x}\cos(\alpha x)\;-\;\alpha(3/4-\alpha^2)\;\;e^{-1/2\,x}\sin(\alpha x)$$

which – amazingly – boils down to:

$$e^{-1/2\,x}\cos(\alpha x)$$

Now – what does this new curve look like?



The blue line is the Kappa$_3$ function and the red line is its first derivative. When *x* is positive, the functions look very much like the hyperbolic trig functions but for *x* negative, they oscillate with ever increasing amplitude.

Since this curve is one of the solutions to the equation

$$\frac{d^3y}{dx^3}\;=\;y$$

it traces the path of a particle whose *rate of change of acceleration* is equal to its displacement. I doubt if it has must application but you never know!

The Kappa$_4$ function

The general solution to the equation $\frac{d^4 y}{dx^4} = y$ has the form:

$$f(x) = A e^{x} + B e^{ix} + C e^{i^2 x} + D e^{i^3 x}$$

and it is easy to see that both the hyperbolic and the standard trig functions are included. I shall define the Kappa$_4$ function as follows:

$$K_4(x) = \frac{e^{x} + e^{ix} + e^{i^2 x} + e^{i^3 x}}{4}$$

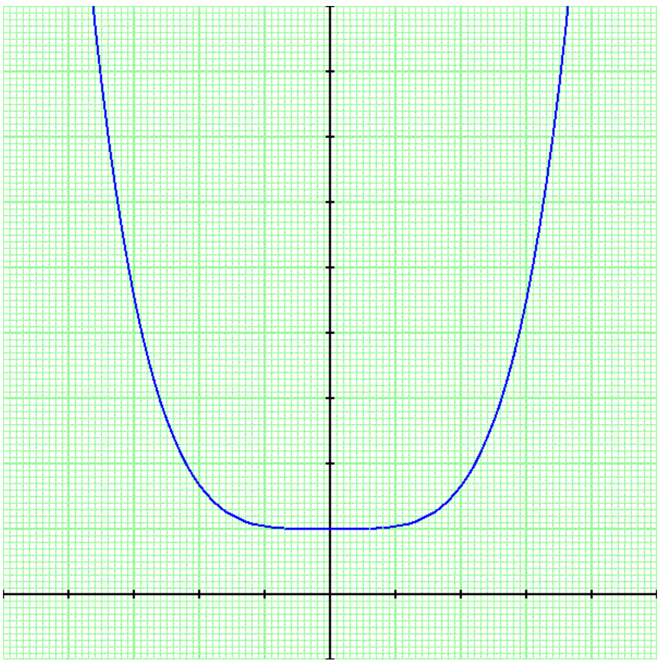
(where $\kappa_4 = i$) and its expansion is:

$$K_4(x) = 1 + \frac{1}{4!}x^4 + \frac{1}{8!}x^8 + \frac{1}{12!}x^{12} + \ldots$$

It can also be expressed in the following way:

$$K_4(x) = \frac{\cosh(x) + \cos(x)}{2}$$

and it looks like this:



Because it is an even function, it is a lot less interesting than the Kappa$_3$ function.

Is is evident that there is a whole series of Kappa functions. For example: Kappa$_8$ can be expressed as

$$K_8(x) = \frac{\cosh(x) + \cos(x) + 2\cosh(x/\sqrt{2})\cos(x/\sqrt{2})}{4}$$

and looks similar to K_4 but with an even longer flat section.

I expect that the odd Kappa functions are quite complicated to express in terms of the standard functions but that they all behave in a similar way to K_3.