# Fibonacci numbers

A general Fibonacci series is characterised by the fact that $F_n = F_{n-1} + F_{n-2}$.

Consider the series 1, $a$, $a^2$, $a^3$, $a^4$ ... This will be a Fibonacci series if, and only if $a^2 = a + 1$. The solution to this quadratic equation is $a = (1 + \sqrt{5})/2$ which is the golden ratio 1.618033...

This number has some remarkable properties; in particular, all powers of $a$ can be simplified as follows:

$$
\begin{aligned}
a^2 &= a + 1 \\
a^3 &= a^2 + a = 2a + 1 \\
a^4 &= a^3 + a^2 = 3a + 2 \\
a^5 &= a^4 + a^3 = 5a + 3 \\
a^6 &= a^5 + a^4 = 8a + 5
\end{aligned}
$$

etc.

It is interesting to see the the natural Fibonacci numbers appearing in this list

It is possible to extend this list backwards to include negative indices as follows:

$$
\begin{aligned}
a^{-6} &= a^{-4} - a^{-5} = -8a + 13 \\
a^{-5} &= a^{-3} - a^{-4} = 5a - 8 \\
a^{-4} &= a^{-2} - a^{-3} = -3a + 5 \\
a^{-3} &= a^{-1} - a^{-2} = 2a - 3 \\
a^{-2} &= a^{0} - a^{-1} = -1a + 2 \\
a^{-1} &= a^{1} - a^{0} = 1a - 1 \\
a^{0} &= 0a + 1 \qquad 1 \\
a^{1} &= 1\,a + 0 \\
a^{2} &= 1\,a + 1 \\
a^3 &= a^2 + a = 2a + 1 \\
a^4 &= a^3 + a^2 = 3a + 2 \\
a^5 &= a^4 + a^3 = 5a + 3 \\
a^6 &= a^5 + a^4 = 8a + 5
\end{aligned}
$$

etc.

The natural Fibonacci numbers are as follows:

$$
\begin{aligned}
\boldsymbol{F_0} &= 0 \\
\boldsymbol{F_1} &= 1 \\
\boldsymbol{F_2} &= 1 \\
\boldsymbol{F_3} &= 2 \\
\boldsymbol{F_4} &= 3 \\
\boldsymbol{F_5} &= 5 \\
\boldsymbol{F_6} &= 8
\end{aligned}
$$

etc.

It would be nice if we had a formula for the $n$th Fibonacci number. By inspecting the above list of powers of the golden ratio, we can see that $a^n = \boldsymbol{F_n}a + \boldsymbol{F_{n-1}}$. This gives us a formula for $\boldsymbol{F_n}$ in terms of $\boldsymbol{F_{n-1}}$, namely:

$$\boldsymbol{F_n} = (a^n - \boldsymbol{F_{n-1}}) / a$$

or

$$\boldsymbol{F_n} = a^{n-1} - \boldsymbol{F_{n-1}} / a$$

Let us write down a list of the Fibonacci numbers in terms of $a$ each time using the above formula to generate the next number:

$F_0 = 0$ (our starting point)
$F_1 = a^0 - 0$
$F_2 = a - 1/a = a^1 - a^{-1}$ ( = 1 )
$F_3 = a^2 - (a^1 - a^{-1})/a = a^2 - a^0 + a^{-2}$ ( = 2 )
$F_4 = a^3 - (a^2 - a^0 + a^{-2})/a = a^3 - a^1 + a^{-1} - a^{-3}$ ( = 3 )
$F_5 = a^4 - (a^3 - a^1 + a^{-1} - a^{-3})/a = a^4 - a^2 + a^0 - a^{-2} + a^{-4}$ ( = 5 )
etc.

from which it is easy to see that

$$F_n = a^{n-1} - a^{n-3} + a^{n-5} - .... \pm a^{-(n-5)} \pm a^{-(n-3)} \pm a^{-(n-1)}$$

While it is true that this is a perfectly good formula for $F_n$ in terms of $n$ alone, it has to be said that if you want to calculate $F_{100}$ say, you are better off calculating the other 99 numbers first than using this formula!

There is a better way to proceed. If n is even we have

$$a^n = F_n a + F_{n-1}$$

and

$$a^{-n} = - \mathrm{F}n\, a + F_{n+1}$$

If we add these together we get the sum of $F_{n+1}$ and $F_{n-1}$ which isn't a lot of use; but if we subtract them we get

$$a^n - a^{-n} = F_n a + F_{n-1} + F_n a - F_{n+1} = 2F_n a - (F_{n+1} - F_{n-1})$$

But $F_{n+1} - F_{n-1} = F_n$ so

$$a^n - a^{-n} = F_n(2a - 1) = \sqrt{5}\, F_n$$

Hence

$$F_n = \frac{a^n - a^{-n}}{\sqrt{5}}$$

If $n$ is odd, the minus sign should be replaced by a plus sign so the general formula can be written

$$F_n = \frac{a^n - (-a)^{-n}}{\sqrt{5}}$$

This is known as Binet's formula.

Now for $F_{100}$!

$$F_{100} = \frac{a^{100} - a^{-(100)}}{\sqrt{5}} = 3.54 \times 10^{20}$$

It is a remarkable fact that a formula which is riddled with irrational numbers like *a* and √5 can generate an integer for all values of *n*.

We can also ask what values it generates for fractional values of *n*. Since it involves the exponent of a negative number, the result will be a complex number $\boldsymbol{F}_{real} + i\boldsymbol{F}_{imaginary}$.

$$\boldsymbol{F}_n = \frac{a^n - (-1)^{-n} a^{-n}}{\sqrt{5}}$$

Now

$$(-1)^{-n} = \cos \pi n - i \sin \pi n$$

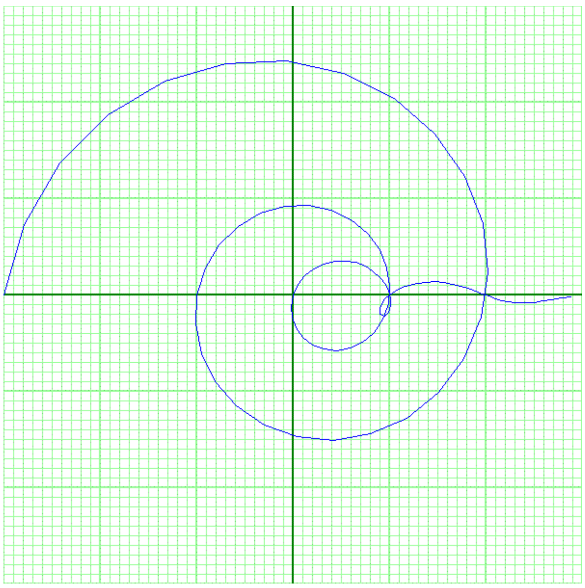
so we can write

$$\boldsymbol{F}_n = \frac{a^n - (\cos \pi n - i \sin \pi n) a^{-n}}{\sqrt{5}}$$

hence

$$\boldsymbol{F}_{real} = \frac{a^n - (\cos \pi n) a^{-n}}{\sqrt{5}} \quad \text{and} \quad \boldsymbol{F}_{imaginary} = \frac{(\sin \pi n) a^{-n}}{\sqrt{5}}$$

A plot of this function is shown below:



It plots *n* from -4 to +4. It crosses the real axis at every integer value of *n*, the crossing points being

-3, 2 -1, 1, 0, 1, 1, 2, 3, ...

which is, amazingly the Fibonacci series!