# Euler's Formula

What justification can we give for the assertion that $e^{i\theta} = \cos\theta + i\sin\theta$ ?

## Justification from derivative

The exponential function can be defined as that function which is equal to its own derivative.

If we think of the function as a function of time, then a particle whose position along the *x* axis obeys the relation $x = e^t$ will have a velocity *v* which is at all times equal to *x*. Such a particle will move along the *x* axis at an exponentially increasing rate.

If the particle obeys the relation $x = e^{-t}$ it will have a velocity *v* which is at all times equal to -*x*. Such a particle will move along the *x* axis in the negative direction at an exponentially decreasing rate.

Suppose now that the particle can move in the complex plane obeying the relation $z = e^{it}$. We shall assume that the velocity of the particle is obtained by using the standard rules for differentiating an expression eg: $v = ie^{it} = iz$. Since the operation of multiplying a complex number by *i* is to turn it through a right angle, we can interpret the motion of this particle as one whose velocity is always equal to $|z|$ and at right angles to *z*. Such a particle must move in a circle at constant speed. Since $z = 1$ when $t = 0$, the circle will be a unit circle and the velocity will be unit also.

Such a motion is described by the parametric equation $x = \cos t$ ; $y = \sin t$ hence $e^{it} = \cos t + i\sin t$

## Justification from the multiplication law

Let us assume that the expression of the form $k e^{ip}$ actually represents a complex number. Our task is to find an interpretation of *k* and *p* which is consistent with the laws of arithmetic as applied to complex numbers.

When two complex numbers $u = r\cos\theta + ir\sin\theta$ and $v = s\cos\phi + is\sin\phi$ are multiplied together, we get $uv = (r\cos\theta s\cos\phi - r\sin\theta s\sin\phi) + i(r\cos\theta s\sin\phi + r\sin\theta s\cos\phi)$.

Amazingly, we can recognise the trigonometrical expressions for the addition of angles within this expression which can therefore be simplified to $uv = rs(\cos(\theta + \phi) + i\sin(\theta + \phi))$.

It follows therefore that when two complex numbers are multiplied together, the moduli ( *r* and *s*) are multiplied and the arguments ($\theta$ and $\varphi$) are added.

Now if we use the ordinary rules of arithmetic to multiply two complex numbers of the form $u = ke^{ip}$ and $v = le^{iq}$ we obtain $uv = kl\,e^{i(p+q)}$. Immediately we notice that the variables *k* and *l* are multiplied together while the variables *p* and *q* are added. It is this that justifies the interpretation that *k* and *l* are the moduli of the two numbers and that *p* and *q* are the arguments.

It follows therefore that the assertion that $re^{i\theta} = r(\cos\theta + i\sin\theta)$ is consistent with the rule for the multiplication of complex numbers.

## **Justification from logarithms**

What interpretation can we give to the expression log(*z*)?

Suppose we start with two complex numbers *u* and *v*. We know that because of the fundamental nature of the logarithm function, log *u* plus log *v* must be equal to the logarithm of the product *uv*.

This strongly suggests that the logarithm of a complex number is in fact its argument. (When complex numbers are multiplied together, the arguments are added)

This cannot be the whole story, however, because we need to be able to recover the original (complex) number by the reverse process of exponentiation. This means that the logarithm of a complex number must itself be complex. In addition, the modulus of the number must play a part in the logarithm as well.

Let us suppose that $\log(r\|\theta) = f(r,\theta) + i\,g(r,\theta)$ where *f* and *g* are (real) functions of *r* and $\theta$ (and where $r\|\theta$ is a complex number of modulus *r* and argument $\theta$.) We require that $e^{f(r,\theta)+ig(r,\theta)} = e^{f(r,\theta)}e^{ig(r,\theta)} = r(\cos\theta + i\sin\theta)$. Since $e^{f(r,\theta)}$ is a real number, it follows that $e^{ig(r,\theta)}$ is a complex one. It seems reasonable therefore to equate the first expression with *r* and the second with $(\cos\theta + i\sin\theta)$. If this is the case then $e^{f(r,\theta)} = r$ (in which case $f(r,\theta) = \log r$ ) and $e^{ig(r,\theta)} = \cos\theta + i\sin\theta$ (which implies that *g* is not a function of *r*). Taking our cue from our earlier suggestion that the logarithm is in fact its argument, let us suppose that $g(r,\theta)$ is in fact nothing other than $\theta$ we arrive at the suggestion that $\log(r\|\theta) = \log r + i\theta$

Does this check out? $\log u + \log v = \log r + i\theta + \log s + i\phi = \log(rs) + i(\theta + \phi)$

Now $\log(uv) = \log(rs\|\theta + \phi) = \log(rs) + i(\theta + \phi)$ so that works out all right.

What about our assumption that $g(r,\theta) = \theta$? This implies immediately that $e^{i\theta} = \cos\theta + i\sin\theta$