Cyclic numbers

Introduction

The number 142857 is quite remarkable. When added to itself repeatedly, it generates the following sequence:

1	142857
2	285714
3	428571
4	571428
5	714285
6	857142
7	**999999**

It seems astonishing that the same digits should reappear over and over again and then suddenly crystallise into a series of nines – but that behaviour gives us the clue that we need. 999999 is divisible by 7. Perhaps if we can find other similar numbers, we can discover more cyclic numbers.

For example, the number 99 is divisible by 11. This generates the following sequence:

1	09
2	18
3	27
4	36
5	45
6	54
7	63
8	72
9	81
10	90
11	**99**

There is something remarkable about this list too. Each of the 5 pairs of digits are cycled eg 27 and 72, 09 and 90 etc.

Lets define some terms. Let us call the first number in the series the **initiate** and number of nines in the final number the **order** of the sequence. The first sequence above has order 6 and the second sequence, order 2. Let us call the number of independent digit sequences the **multiplicity** of the sequence. The multiplicity of the first sequence is 1 while the second has multiplicity 5. Lastly, let us refer to the length of the sequence as the **modality**. The first sequence has modality 7 and the second, modality 11.

From a consideration of just these two sequences, it seems probable that the modality must be equal to the product of the order and the multiplicity plus 1. It also seems likely that the modality must be prime.

In what follows we shall test these two theories and try to discover some more cyclic numbers including numbers in bases other than 10.

Dividing unity

Consider the process of dividing unity by a prime number (which is not a factor of the number base we are working in). For example, let us divide 1 by 7. The sum goes like this:

10 over 7 = 1 remainder 3
30 over 7 = 4 remainder 2
20 over 7 = 2 remainder 6
60 over 7 = 8 remainder 4
40 over 7 = 5 remainder 5
50 over 7 = 7 remainder 1
10 over 7 = 1 remainder 3
etc. etc.

From which we see that 1/7 = 0.142857 recurring. The recurring sequence has order 6 (ie is 6 digits long). It follows that 1000000/7 = 142857.142857 recurring = 142857 + 1/7. Multiplying by 7 we find that 1000000 = 142857 × 7 + 1 or 142857 × 7 = 999999.

All this strongly suggests that we can search for cyclic numbers by dividing 1 by various prime numbers and looking at the pattern of repeating decimals that we obtain. Here is a short list.

2	(factor of 10)
3	0.3 recurring
5	(factor of 10)
7	0.142857 recurring
11	0.09 recurring
13	0.076923 recurring
17	0.0588235294117647 recurring

We have seen how primes 7 and 11 give rise to cycles with order and multiplicity 6; 1 and 2; 5 respectively. The mode 13 cycle has order 6 and multiplicity 2 while mode 17 has order 16 and multiplicity 1. This gives us the next true cyclic number in base 10!

0588235294117647 = 9999999999999999 ÷ 17

(The mode 3 cycle is a degenerate cycle having order 1, multiplicity 2.)

Predicting order and multiplicity

Given a base number b and a modal prime p, what determines the order and the multiplicity of the sequence that results?

Looking back at the process whereby we divided 1 by 7, we see that the remainders are crucial. At each step we take the previous remainder, multiply it by the base and divide by the prime to obtain the next remainder. In mathematical terms

new remainder = {old remainder × base} MOD mode

or, more succinctly $$r_1 = |r_0 b|_p$$

As soon as this generates a remainder which we have had before, the cycle is complete. Obviously this must happen sooner or later and since the remainder can never be zero (because the mode and the base have no common factor), the order of the cycle must be less than the mode.

But why do some cycles have multiplicities of 1 and others not?

To answer this question we must examine the process of calculating the new remainders from the old ones in more detail.

The next remainder in the sequence is going to be

$$r_2 \;=\; |\, r_1 b \,|_p \;=\; ||\, r_0 b \,|_p \, b \,|_p = |\, r_0 b^2 \,|_p \;=\; r_0 |\, b^2 \,|_p$$

and, indeed, the n[th] remainder will be

$$r_n \;=\; r_0 |\, b^n \,|_p$$

Now the cycle is complete when $r_n = r_0$, ie when

$$|\, b^n \,|_p \;=\; 1$$

or, to put it another way, when $b^n - 1$ is divisible by p.

Now, as we have seen, the cycle has to complete before $n = p$. Indeed it will always complete when $n = p$ - 1. This implies that

$$|\, b^{p-1} \,|_p \;=\; 1$$

a theorem which is known as Fermat's Little Theorem.

Now if it should happen that $|\, b^n \,|_p \;=\; 1$ for some smaller value of n, the cycle will complete early and the sequence will have a multiplicity greater than 1. Suppose that $|\, b^m \,|_p \;=\; 1$ where m is a factor of n. Whatever remainder we start with, we will always complete the cycle after m processes. What this means is that all the cycles must have the same length. It is not possible, for example, for a prime like 13 to generate one cycle with order 6 and two with order 3.

The next question to ask is – what happens if the mode is not prime?

The first thing to say is that if the mode shares any factors with the base, these factors soon 'divide out' leaving just the digits associated with the remaining prime. For example, 1/14 = 0.07[142857] and 1/35 = 0.02857[142857] etc.

Of more interest is what happens when the mode has two (or more) prime factors which are not shared with the base. eg 1/21=0.047619 recurring which has order 6. When add the number 047619 to itself repeatedly, this is what we get:

1	047619			
2		095238		
3			142857	
4	190476			
5		238095		
6			285714	
7				333333
8		380952		
9			428571	
10	476190			
11		523809		
12			571428	
13	619057			
14				666666
15			714285	
16	761904			
17		809523		
18			857142	
19	904761			
20		952380		
21	-------------------- **999999** --------------------			

We obtain the digits 047619 six times, the digits 095238 six times and the digits 142857 six times but we also come across the numbers 333333 and 666666. You could say that the mode 21 generates two **groups** of cycles – one of order 6 with a multiplicity of 3 and a degenerate cycle of order 1, multiplicity 2[1]. (Note that 6×3 + 1×2 is one less than 21)

As another example, the mode 77 generates two groups, one with order 6, multiplicity of 11 and the other with order 2 multiplicity 5. (Note that 6×11 + 2×5 is one less than 77)

Summary so far

We have therefore established (not very rigorously) the following facts:

Cyclic numbers in any **base** *b* are formed of the recurring digits in the basal expansion of the reciprocal of any number *p* – called the **mode**. The smallest number which can be formed from this sequence of digits is called the **initiate**.

If the mode shares any factors with the base, these make no difference to the initiate so we only need consider modal numbers which are co-prime with the base.

When the mode *p* is prime, the recurring digits generate a single **group** of cyclic digits whose **order** (ie the number of digits in the recurring group) is one less than the mode. This is because, by Fermat's little theorem, the number $b^{(p-1)} – 1$ is always divisible by *p*. It can, however, happen that a smaller number $b^q – 1$ is also divisible by *p* where *q* is a factor of $p – 1$. In this case the group contains more than one different sets of digits which are mixed together but all the sets must have the same order. The number of different sets is called the multiplicity of the cycle. It follows that:

mode = order × multiplicity + 1

When the mode is composite, the resulting sequence of numbers may contain more than one group of cycles with different orders and multiplicity. It remains true, however, that:

mode = SUM[order × multiplicity] + 1

The most interesting initiates are those which generate a single group whose multiplicity is 1. This means using a mode *p* which is prime and where $b^q – 1$ is not divisible by *p* (*q* being any factor of *p* - 1)

How do cyclic numbers actually work?

Now we know how to generate cyclic numbers, lets look a bit more as to how and why they actually work.

We saw earlier that the classic example – 142857 – is generated by calculating the reciprocal of 7. This is 0.14285714285714...

Lets make a list of all the integers divided by 7

1/7 = 0.14285714285714..
2/7 = 0.28571428571428...
3/7 = 0.42857142857142...
4/7 = 0.57142857142857...
5/7 = 0.71428571428571...
6/7 = 0.85714285714285...
7/7 = 0.99999999999999...

1 The cycle 333333, 666666, 999999 has order 1 not 6 because there is only 1 way of pernutating the digits of each number. It has multiplicity 2 because there are 2 sets of digits comprising the cycle, namely 333333 and 666666. We do not count the final total of 999999 because this is the terminus of all the cycles.

The fundamental question is – why do the same digits occur again and again in the same sequence?

Let us remind ourselves how we divide 1 by 7:

10 over 7 = 1 remainder 3
30 over 7 = 4 remainder 2
20 over 7 = 2 remainder 6
60 over 7 = 8 remainder 4
40 over 7 = 5 remainder 5
50 over 7 = 7 remainder 1
10 over 7 = 1 remainder 3
etc. etc.

It is obvious that here are only 6 possible remainders (0 is not allowed as 7 does not divide into 10 or any of its powers) so after 6 processes, we must generate a remainder that we have had before – hence the recurring nature of the result – but the really crucial point is that whatever number (less than 7) which we start with, we plunge straight into the repeating cycle at some point or other. This means that all the numbers from 1/7 to 6/7 must generate exactly the same repeating cycle, only starting at a different point. What this means is that the first 6 digits of the expansion must contain the same 6 numbers.

So if we multiply the list by 1000000 we get

1000000/7 = 142857 + 1/7
2000000/7 = 285714 + 2/7
3000000/7 = 428571 + 3/7
4000000/7 = 571428 + 4/7
5000000/7 = 714285 + 5/7
6000000/7 = 857142 + 6/7
7000000/7 = 999999 + 7/7

and subtract the fractional bits

(1000000 - 1)/7 = 1 × 999999/7 = 142857
(2000000 - 2)/7 = 2 × 999999/7 = 285714
(3000000 - 3)/7 = 3 × 999999/7 = 428571
(4000000 - 4)/7 = 4 × 999999/7 = 571428
(5000000 - 5)/7 = 5 × 999999/7 = 714285
(6000000 - 6)/7 = 6 × 999999/7 = 857142
(7000000 - 7)/7 = 7 × 999999/7 = 999999

we can see exactly how the sequence is generated.

It is also of interest to see how a multiple sequence is generated. Lets try mode 13

1/13 = 0.07692307692307...
2/13 = 0.15384615384615...
3/13 = 0.23079623079623...
etc.etc.

This time, owing to the fact that 999999 happens to be divisible by 13, the 12 possible remainders split into two groups of 6.

Determining the order and multiplicity of a cycle

The question now arises – is there any general way of predicting the orders and the multiplicities of the cycles generated by a given mode? Could we have predicted, for example, that the mode 13 cycle would have a multiplicity of 2? I think the answer to this is no. Fermat's Little theorem guarantees that the number 999,999,999,999 will be divisible by 13 but it is only chance that causes the number 999,999 to be divisible by 13 also.

It would be better, perhaps, to ask a slightly different question. What cycles are there of order *s*?

Now every cycle of order *s* ends with a number of the form 9999...9 containing *s* 9's. All we have to do then is to find all the factors of this number, each one of which will generate a cycle of order *s*. For example. Lets see if we can find a cycle of order 5.

The factors of 9999 are 3 × 3 × 41 × 271. This means that there will be two cycles of order 5 whose initiates are 99999/41 = 02439 and 99999/271 = 00369. Both these cycles will have a multiplicity of 1 since there is no smaller string of 9's which divides by either 41 or 271. The mode 41 cycle will therefore have order 5 and multiplicity 8 (because 5 × 8 = 41 – 1) and the mode 271 cycle will have order 5 and multiplicity 54.

Here is a list of all the cycles up to order 16. Note that the prime factors 2, 3 and 5 are degenerate and are not included. Also, once a prime factor has been used (eg 11, 37) it is ignored thereafter.)

Order	Terminator	Prime factor	Initiate	Multiplicity
2	99	11	09	5
3	999	37	027	12
4	9,999	101	0099	20
5	99,999	41	02439	8
5	99,999	271	00369	54
6	999,999	7	142857	1
6	999,999	13	076923	2
7	9,999,999	239	0041841	34
7	9,999,999	4649	0002151	664
8	99,999,999	73	01369863	9
8	99,999,999	137	00729927	17
9	999,999,999	333667	000002997	37074
10	9,999,999,999	9091	0001099989	909
11	99,999,999,999	21649	---	1968
11	99,999,999,999	513239	---	4658
12	999,999,999,999	9901	---	825
13	9,999,999,999,999	53	---	4
13	9,999,999,999,999	79	---	6
13	9,999,999,999,999	265371653	---	20413204
14	99,999,999,999,999	909091	---	64935
15	999,999,999,999,999	2906161	---	193744
16	9,999,999,999,999,999	17	0588235294117647	1
16	9,999,999,999,999,999	5882353	---	367647

In compiling this list it occurs to me that if a number with *s* 9's (i.e. a number of the form 10^s – 1) is divisible by a prime *p* then *p* – 1 must be divisible by *s*. For example, since 99,999 is divisible by the prime 41, then there will be a cycle of order *s* = 5. Now since whenever the mode is prime the order × multiplicity = mode – 1, and since the multiplicity must be a whole number, it follows that the mode – 1 must be divisible by the order i.e. $p – 1$ must be divisible by s. It is by no means obvious that this is necessarily the case.

Putting this theorem into mathematical terms, if $10^s - 1$ is divisible by p then $p – 1$ must be divisible by s. (Note that this is true for all s. s does not have to be prime.)

I can prove that this statement is consistent with Fermat's Little Theorem but I have been unable to deduce it.

Cyclic numbers in other bases

Lets see what we can find in base 9.

For the mode $p = 5$, we need to calculate the number 8888/5 which is 1717. This generates the sequence: 1717, 3535, 5353, 7171, 8888 which is of order 2, multiplicity 2. The doubling of the digits suggests that 88 is also divisible by 5 so the number 17 will generate a similar sequence.

Here is a list of cyclic sequences in bases from 2 to 16 using primes up to 13. Primes which produce degenerate sequences are omitted.

True cyclic numbers (i.e. those which generate sequences with a multiplicity of 1) are highlighted in yellow.

Base	Mode	Initiate	Ord	Mult	Sequence
2	3	01	2	1	01, 10
2	5	0101	4	1	0011, 00110, 1001, 1100
2	7	001001	3	2	001001, 10010, 11011, 100100, 101101, 110110
2	11	0001011101	10	1	0001011101, 0010111010, 0100010111, 0101110100, 111010001, 1000101110, 1010001011, 1011101000, 1101000101, 1110100010
2	13	000100111011	12	1	000100111011, 001001110110, 001110110001, 010011101100, 011000100111, 011101100010, 100010011101, 100111011000, 101100010011, 110001001110, 110110001001, 1110100010
3	5	0121	4	1	0121, 1012, 1210, 2101,2222
3	7	010212	6	1	010212, 02101, 102120,120102, 201021, 212010
3	11	00211	5	2	00211, 01122, 02110, 10021, 11002, 11220, 12201, 20112, 21100, 22011
3	13	002	3	4	002, 011, 020, 022, 101, 110, 112, 121, 200,202, 211, 220
4	5	03	2	2	03, 12, 21, 30
4	7	021	3	2	021, 102, 123, 210, 231, 312
4	11	01131	5	2	01131, 02322,10113, 11310, 13101, 20232, 22023, 23220, 31011, 32202
4	13	010323	6	2	010323, 021312, 032301, 103230, 1210213, 131202, 202131, 213120, 230103, 301302,

					312021, 323010
5	3	13	2	1	13, 31
5	7	032412	6	1	032412, 120324, 203241, 241203, 324120, 412032
5	11	02114	5	2	02114, 04233, 11402, 14021, 21140, 23304, 30423, 33042, 40211, 42330
5	13	0143	4	3	0143, 0341, 1034, 1232, 1430, 2123, 2321, 3014, 3212, 3410, 4103, 4301
6	7	05	2	3	05, 14, 23, 32, 41, 50
6	11	0313452421	10	1	0313452421, 1031345242, 1345242103, etc.
6	13	024340531215	12	1	024340531215, 053121502434, 121502434053 etc.
7	5	1254	4	1	1254, 2541, 4125, 5412
7	11	0431162355	10	1	0431162355, 1162355043, 1623550431 etc.
7	13	035245631421	12	1	035245631421, 103524563142, 142103524563 etc.
8	3	25	2	1	25, 52
8	5	1463	4	1	1463, 3146, 4631, 6314
8	11	0564272135	10	1	0564272135, 1350564272, 2135056427 etc.
8	13	0473	4	3	0473, 1166, 1661, 2354, 3047, 3542, 4235, 4730, 5423,6116, 6611, 7304
9	5	17	2	2	17, 35, 53, 71
9	7	125	3	2	125,251, 376, 512, 637, 763
9	11	07342	5	2	07324, 15648, 24073, 32407, 40732, 48156, 56481, 64815, 73240, 81564
9	13	062	3	4	062, 134, 206, 268, 341, 413, 475, 547, 620, 682, 754, 826
10	7	142857	6	1	142857, 285714, 428571, 571428, 714285, 857142
10	11	09	2	5	09, 18, 27, 36, 45, 54, 63, 72, 81, 90
10	13	076923	5	2	076923, 153846, 230769, 307692, 384615, 461538, 538461, 614384, 692307, 769230, 846153, 923076
11	7	163	3	2	163, 316, 479, 631, 794, 947
11	13	093425A17685	12	1	093425A17685, 17685093425A, 25A176850934 etc.
12	5	2497	4	1	2497, 4972, 7249, 9724
12	7	186A35	6	1	186A35, 35186A, 5186A3, 6A3518, 86A351
12	13	0B	2	6	0B, 1A, 29, 38, 47, 56, 65, 74, 83, 92, A1, B0
13	5	27A5	4	1	27A5, 527A, 7A52, A527
13	7	1B	2	3	1B, 39, 57, 75, 93, B1

13	11	12495BA837	10	1	12495BA837, 2495BA8371, 3712495BA8 etc.
14	3	49	2	1	49, 94
14	5	2B	2	2	2B, 58, 85, B2
14	11	13B65	5	2	13B65, 278CA, 3B651, 513B6, 6513B, 78CA2, 8CA27, A278C, B6513, CA278
15	11	156C4	5	2	156C4, 2AD98, 4156C, 56C41, 6C415, 82AD9, 982AD, AD982, C4156, D982A
15	13	124936DCA5B8	12	1	124936DCA5B8, 24936DCA5B81, 36DCA5B81249 etc.
16	7	249	3	2	249, 492, 6DB, 924, B6D, DB6
16	11	1745D	5	2	1745D, 2E8BA, 45D17, 5D174, 745D1, 8BA2E, A2E8B, BA2E8, D1745, E8BA2
16	13	13B	3	4	13B, 276, 3B1, 4EC, 627, 762, 89D, 9D8, B13, C4E, D89, EC4

But lets not forget my favourite cycle in base 10 with order 16:

0588235294117647
1176470588235294
1764705882352941
2352941176470588
2941176470588235
3529411764705882
4117647058823529
4705882352941176
5294117647058823
5882352941176470
6470588235294117
7058823529411764
7647058823529411
8235294117647058
8823529411764705
9411764705882352
9999999999999999