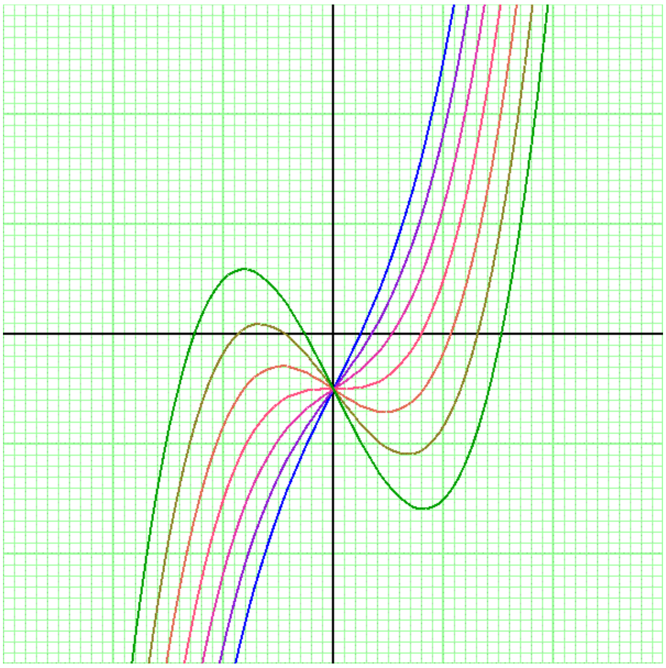
Cubic Equations

The general equation $ax^3 + bx^2 + cx + d = 0$ can always be reduced to the form $\boldsymbol{x^3 = px + q}$ by a linear substitution..



The graph of $y = x^3 – px – q$ has the following characteristics: Obviously $-q$ is the point at which the curve cuts the Y axis; The lack of an x^2 term implies that the point of inflexion is at $x = 0$; p determines how 'wiggly' the curve is. When $p > 0$, the curve has a reflex shape; when $p = 0$, the inflexion is horizontal and when $p < 0$, the gradient of the curve is always positive.

Scipione del Ferro's solution

Scipione del Ferro, (c 1465 - 1526), professor of Mathematics at the university of Bologna found a remarkable solution to the equation $x^3 = px + q$ which was later published by Geronimo Cardano (1501 – 1576) in 1545.

$$x = \sqrt[3]{(q/2 + \sqrt{D})} + \sqrt[3]{(q/2 - \sqrt{D})}$$

where $$D = (q/2)^2 – (p/3)^3$$

We shall consider the solution in several stages. We shall also assume (without loss of generality) that q is positive.

Case 1: First: when $p<0$, as we have noted, the curve is not reflex and therefore has only one (real) solution. This follows from del Ferro's formula because **D** will always be positive, $\sqrt{\mathbf{D}}$ will be real and although it has two values, they both lead to the same value for x.

e.g. if $x^3 = -3x + 4$, $\mathbf{D} = 5$ and $x = \sqrt[3]{(2 + \sqrt{5})} + \sqrt[3]{(2 - \sqrt{5})} = 1$

Case 2: When $p = 0$, del Ferro's solution reduces to $x = \sqrt[3]{(q/2 + q/2)} + \sqrt[3]{(q/2 – q/2)} = \sqrt[3]{(q)}$

Case 3: Next we consider the case when $p>0$ but $(q/2)^2 > (p/3)^3$ so **D** is still positive. The curve now has a ‘wiggle’ but the loop does not reach the X axis so the equation still only has one solution.

e.g. if $x^3 = 3x + 4$, **D** = 3 and $x = \sqrt[3]{2 + \sqrt{3}} + \sqrt[3]{2 - \sqrt{3}} = 2.196$

Case 4: Now consider what happens when p is such that $(q/2)^2 = (p/3)^3$ so **D** is zero. This is the moment when the ‘wiggle’ becomes sufficiently large so that it just touches the X axis. At this point, the equation has two solutions. Del Ferro’s formula gives

e.g. if $x^3 = 4.76x + 4$, **D** = 0 and $x = \sqrt[3]{2} + \sqrt[3]{2} = 2.52$

but there is a second solution equal to -1.25. Where is it? In order to find it we must consider the last and most interesting case, when **D** is negative.

Case 5: Now $(q/2)^2 < (p/3)^3$ and $\sqrt{\mathbf{D}}$ is imaginary.

e.g. if $x^3 = 6x + 4$, **D** = -4 and $x = \sqrt[3]{2 + \sqrt{-4}} + \sqrt[3]{2 - \sqrt{-4}}$

Naturally enough, Cardano was at a loss to know how to deal with the square roots indicated but he know that the equation has three perfectly good roots and that del Ferro’s formula worked for all the other cases – so why not this one?

Nowadays, we might proceed as follows. First we write the solution in terms of complex numbers.

$$x = \sqrt[3]{2 + 2\mathbf{i}} + \sqrt[3]{2 - 2\mathbf{i}}$$

We note that the two complex numbers are conjugates (~) of one another. Using the fact that $\sqrt[3]{-D} = -\sqrt[3]{D}$ we have

$$x = \sqrt[3]{2 + 2\mathbf{i}} + \sim\sqrt[3]{2 + 2\mathbf{i}}$$

This is, of course, a real number so

$$x = 2 \,.\, \mathbf{real}[\, \sqrt[3]{2 + 2\mathbf{i}} \,]$$

and since there are three cube roots, the equation has three solutions.: 2.732, -2 and -0.732.

Case 4 again: Now what about that missing solution? In order to find it, we must realise that even the cube root of a positive number has complex roots too. In our example, $\sqrt[3]{2}$ is not just 1.26, it is also (-0.63 + 1.09**i**) and (-0.63 - 1.09**i**). In calculating $\sqrt[3]{2} + \sqrt[3]{2}$, we must take both possibilities into account which leads to the missing solution of -1.26. (A similar argument will lead to the missing (complex) solutions of cases 1-3.)