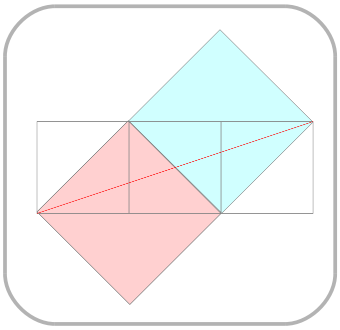
Arctan relationships

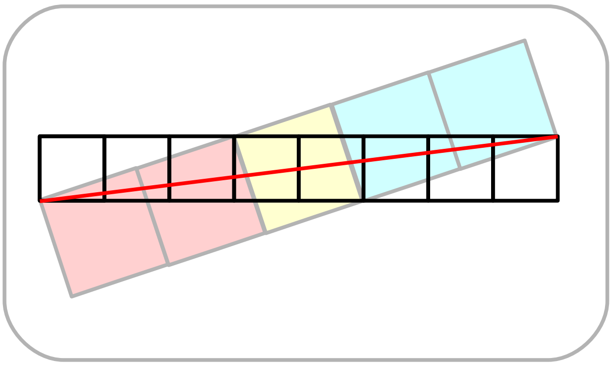
atan(1) = atan(1/2) + atan(1/3)   or   {1} = {2} + {3}



This works because having constructed the red square on the first two unit cells, it is clear that the corner of the blue square must coincide with the corner of the third square.

We derive the desired relation by considering the angles on each side of the red line

atan(1/3) = atan(1/5) + atan(1/8) or   {3} = {5} + {8}

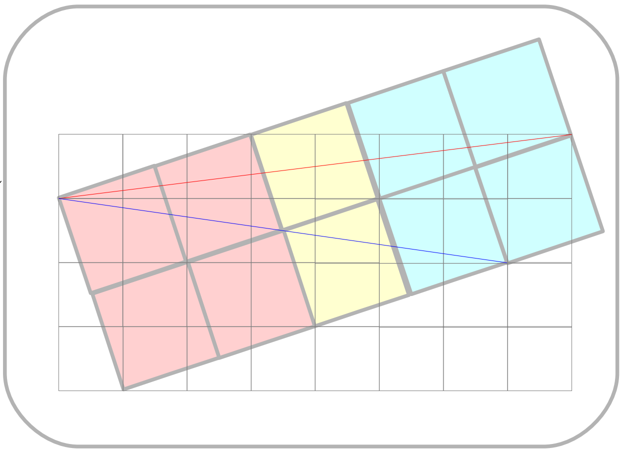


It is not immediately obvious that this arrangement works. Having constructed the two red squares, we must demonstrate that the corner of the yellow square coincides with the end of the fifth unit cell.

The large squares have sides equal to √10 / 2, therefore its diagonal is equal to √5.

But this is exactly the same as the length of the diagonal of a 2 × 1 rectangle.

This does not prove that the points are coincident though. To do this we must look at the wider picture



Since the red squares are inclined at an angle of atan(1/3), a 2 × 2 array of squares will fit into a 4 × 4 array of cells. By symmetry the centres must be coincident. This is sufficient to prove that the corner of the yellow square must lie on a corner of a unit cell and similarly for the corner of the blue square.

In fact, once you have superimposed two grids like this, you can generate any number of formulae involving atan(1/3)

eg by considering the blue line we find that

atan(1/2) = atan(1/3) + atan(1/7)   or   {2} = {3} + {7}

and by considering grids at other angles, a huge number of relationships can be generated.