The Twins Paradox

Arthur has just read Linton's article explaining the famous Twins Paradox which his sister Betty gave him but he is not convinced.

A: All these graphs – its just too confusing! What with Doppler shifts, time dilation and length contraction – isn't there a simpler explanation?

B: Well, it depends what you mean by simpler. Some people like real figures and pictures, others are happier with a more abstract approach.

A: Well, I am not sure I like the sound of that any better but you might as well give it a try.

B: Okay – but you are going to have to let me start somewhere.

A: What do you mean?

B: Well – Linton just assumed the formulae for length contraction and time dilation. If we are going to start somewhere else, we must make some similar assumptions.

A: Fair enough.

B: First lets start with the concept of space-time. Any *event* (such as your birth or the explosion of a distant supernova) takes place at a certain location and at a certain point in time. For example, you could specify your birth in space time by stating the planet you were born on, the latitude and longitude of your birthplace and the time and date. In general you need three spacial coordinates and one temporal one to fix the location of any event in space-time. In fact we can refer to a set of 4 coordinates (x, y, z, t) as an *event* whether or not anything actually happens there.

A: Yes, I get that

B: Good. But we shall be interested in the relationship between two such sets of coordinates – Albert's and Ludvig's – which are in relative motion. We shall assume that at the instant Albert sets off on his travels, these two coordinate systems coincide. i.e. the event of Albert's departure is (0, 0, 0, 0) in both systems and that Albert moves off in the X direction. What this means is that the Y and Z coordinates of Albert's journey are always zero so we shall forget them from now on.

A: So in Ludvig's system, Albert travels 4 light years in 5 years and then comes back again while in Albert's system, he remains stationary the whole time. Is that right?

B: Yes, on the outward journey; but when Albert turns round he has to abandon his outward coordinate system and step on a different system for the journey home. So there are really *three* coordinate systems to consider. Ludvig can use one system for the whole time but Albert cannot. *This* is why there is an assymmetry in the situation.

A: Yes I see. But can you show that it is Ludvig and not Albert who is older when the brothers are reunited?

B: Yes, but to do this we need to employ some principle which embodies the basic assumption on which Special Relativity rests – namely the constancy of the speed of light.

A: And what might that be?

B: It takes the form of a set of equations which relates the coordinates of an event in one system (x, y, z, t) with the coordinates in another (x', y', z', t') which is in relative motion (along the X axis) with speed v. These equations are:

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - vx/c^2)$$
$$\text{where } \gamma = \frac{1}{\sqrt{(1 - v^2/c^2)}}$$

A: Ah! I can see the length contraction and time dilation factor in there!

B: Absolutely.

A: But how do we use these equations? What do they do?

B: Well, if you know the coordinates of and event in one frame, you can instantly work out its coordinates in another. For example, in Ludvig's frame, Albert's departure is (0, 0, 0, 0) and you can easily verify that when $x = 0$, $y = 0$, $z = 0$ and $t = 0$, then so do x', y', z' and t'.

A: I see. Now according to Ludvig, his brother reaches Alpha Centauri (a distance of 4 light years) 5 years later (travelling at 80% of the speed of light). So $x = 4$, y and z are both zero of course, and $t = 5$. i.e Albert arrives at his destination at (4, 0, 0, 5) in Ludvig's frame. Am, I right?

B: Perfectly. Now work out the coordinates in Albert's frame.

A: Okay. First $\gamma = 1.67$ doesn't it? So $x' = 1.67(4 - 0.8\times5)$ which equals... Oh! Of course. It equals zero doesn't it? Because Albert's X coordinate is always zero in his frame!

B: Correct. But what about his t coordinate?

A: OK. $t' = 1.67(5 - 0.8\times4/c^2)$ But what do I put in for c?

B: Well, since you have been using light-years for the distances and years for the time, c is just 1 light-year per year so put in 1.)

A: I see. Well that comes to $t' = 1.67(5 - 0.8\times4)$ which is – 3 years exactly!

B: Precisely. Albert is 3 years older when he arrives at Alpha Centauri – not 5 years.

A: So if I put $x = 0$ and $t = 10$ into the formula I should be able to show that he is only 6 years old when he gets back home shouldn't I?

B: Hang on a minute..

A: No don't spoil my fun – let me work it out... $t' = 1.67(10 - 0.8\times0)$ which equals – wait a mo! What's gone wrong? That comes to 16.7 years not 6!

B: I tried to tell you.

A: What?

B: You have forgotten what I said earlier. When Albert turns round he has to abandon his outward coordinate system and hop onto another one travelling in the opposite direction. So you can't just plug the new numbers into the old equations.

A: So what equations can we use?

B: Well imagine a kind of mirror image of Albert reflected in a magic mirror on Alpha Centauri. When Albert departs from Earth his image is 8 light years away and as Albert travels towards the star, his image converges on him and they return to Earth together. What we must do is write down the equations which relate Ludvig's system to that of Albert's image.

A: How do we do that?

B: The first thing to note is that Albert's image travels in the opposite direction so v is -0.8 not 0.8. Second we must arrange it so that when $t = 0$ and $x = 8$ (Albert's image is 8 light years from Earth at the start) then $x' = 0$. These equations will do the trick:

$$x' = 1.67((x - 8) + 0.8t)$$
$$t' = 1.67(t + 0.8(x - 8))$$

A: Yes =- I can see that when $t = 0$ and $x = 8$ then $x' = 0$. And I see that you have changed the sign of v.

B: Now put in Ludvig's coordinates for Albert's return i.e $x = 0$ and $t = 10$.

A: Here we go - $t' = 1.67(10 + 0.8\times(0 - 8))$ which is, hang on, yes! It is 6 years!

B: Excellent! Are you any more satisfied?

A: Well, to be honest, not really. I am still finding it difficult to accept that time actually goes more slowly for Albert than it does for Ludvig.

B: That is because you still haven't grasped the subtlety of time dilation. Time does *not* go more slowly for Albert. For him, time proceeds at its normal rate. It is the *distance* which he has to travel which is shorter than he expects. For Ludvig, the distance he travels is the same but (according to Ludvig) Albert's clocks and metabolic rate and everything else are going slow so he takes less *time*.

A: I suppose so.

B: Put it this way. Both Albert and Ludvig travel through space-time from (in Ludvig's frame) (0, 0, 0, 0) to (0, 0, 0, 10). But they take different routes to get there. Ludvig passes through (0, 0, 0, 5) while Albert passes through (4, 0, 0, 5). It is a bit as if Ludvig travels from A to B directly but Albert goes via C. In ordinary space, the distance betweetbut in space-time two points is calculated using Pythagoras' theorem:

$$\text{Distance between two points} = \sqrt{\Delta x^2 \ + \ \Delta y^2 \ + \ \Delta z^2}$$

(where the Δ sign means 'change in')

but in space-time you have to calculate the distance (or 'interval') between two events differently:

$$\text{Interval between two events} = \sqrt{\Delta x^2 \ + \ \Delta y^2 \ + \ \Delta z^2 \ - \ \Delta(ct)^2}$$

Note the crucial minus sign in front of the temporal term.

The interval between (0, 0, 0, 0) to (0, 0, 0, 10) is $10i$ (where i is the square root of minus one. Don't worry about this. Just think of it as an arbitrary number.)

Now the interval between (0, 0, 0, 0) and (4, 0, 0, 5) is $3i$. So is the interval between (4, 0, 0, 5) and (0, 0, 0, 10). So the total interval between (0, 0, 0, 0) and (0, 0, 0, 10) going via (4, 0, 0, 5) is only $6i$. It may seem odd that going the 'long way round' takes a shorter time – but that's relativity for you!