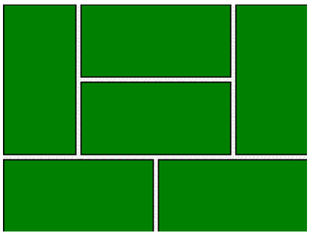
Tatami Tilings and Patio Pavings

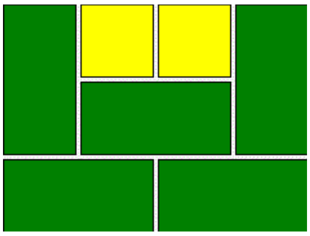
Tatami Tilings

Japanese rooms have traditionally been covered with floor mats with an aspect ratio of 2:1 ('dimers'). For the best *Fu Sui* (the Japanese version of Feng Shui) none of the vertices should be where 4 mats meet – indeed the + crossings look decidedly ugly. Such a tiling is called a Tatami tiling. The smallest room that can be covered with Tatami tiles in a non-trivial way is 4 × 3. Here is an example:



4 × 3 Tatami tiling

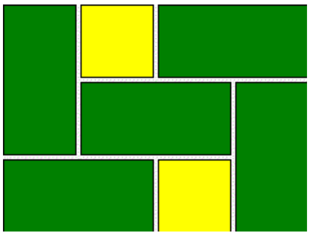
Occasionally, 1 × 1 tiles ('monomers') can be used. If the room has an odd area or if it has a strange shape the use of at least one monomer may be essential. The 4 × 3 room can also be tiled using a pair of monomers instead of one of the dimers in a number of ways, the most elegant of which is:



4 × 3 Tatami tiling with monomers

Indeed it may be considered to have even greater *Fu Sui* than the first example because it avoids the unsightly coincidence of the two shared long edges. It is, in my view however, still not 'perfect' for two reasons. Firstly the two monomers still share a common edge and secondly it can be split into two rectangular sections because of the unsightly horizontal crack which extends from one side to the other.

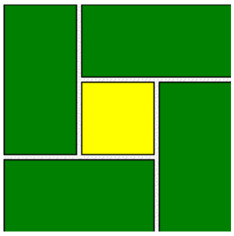
Is there a better Tatami tiling of a 4× 3 room using 5 dimers and 2 monomers? Yes there is:



Perfect 4 × 3 Tatami tiling

This is the one I would choose. It has just the right mix of randomness and symmetry.

Now you might object that this tiling is, in fact worse than the previous one because now there are *two* pairs of tiles which share a common edge. A little thought will convince you that the *only* way of surrounding a monomer with dimers (or other tiles) without butting short edges looks like this:



Perfect 3 × 3 Tatami tiling

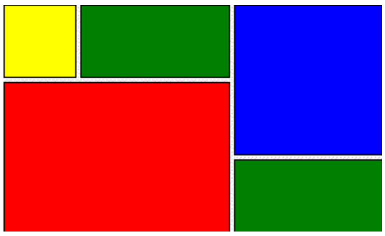
so if we don't allow monomers to butt against dimers, we won't find any perfect tilings. For this reason, this particular configuration is permitted.

Strangely, the mathematical aspects of these tilings do not seem to have been explored until quite recently ('There is an extensive literature on 'Domino' tilings where + crossings are permitted and only dimers are used) but I shall not pursue this field; I wish to explore a slightly different problem which may not have received any attention from serious amateurs at all.

Patio Pavings

A few years ago I had occasion to lay a new patio and my local builders yard supplied me with a pack of paving slabs. On opening the pack I found that it contained a certain number of 3 × 2 pavers (hexamers), a few 2 × 2 pavers (tetramers), several dimers and some monomers. (I forget exactly how many there were of each.) A number of interesting mathematical problems immediately present themselves, the most obvious of which is: given a certain pack of paving slabs, is it possible to pave a certain patio with the same total area with a perfect tatami tiling which includes at least one of each type of slab? (I shall call this a 'perfect patio paving'.)

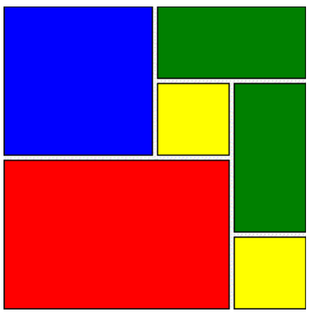
Since the total area of 1 hexamer, I tetramer, 1 dimer and 1 monomer is 13, the smallest rectangular patio which is of interest is 5 × 3 for which we need an extra dimer but it is easy to show that no perfect paving exists for this patio. The best you can do is:



Imperfect 5 × 3 paving

but this has one common edge and one crack.

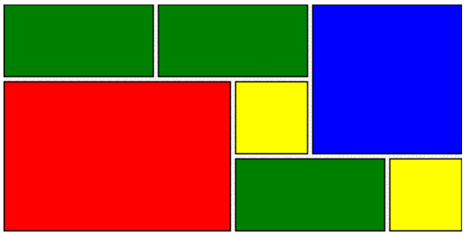
What about a 4 × 4 patio? For this we need 1 hexamer, I tetramer, 2 dimers and 2 monomers. Now we have what looks like our first unique solution:



Perfect 4 × 4 paving

(Note that we are going to allow a dimer to butt against a monomer.)

The next patio up is 6 × 3. The best you can do here is:

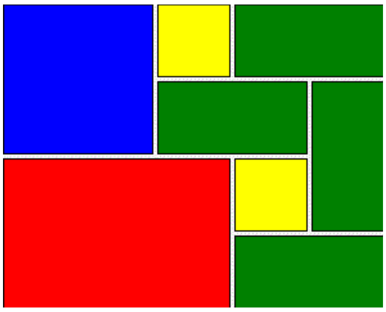


Imperfect 6 × 3 paving

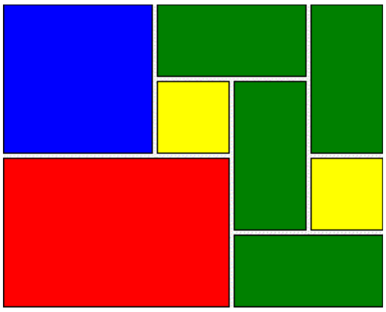
but this is not quite perfect because two of the dimers share a short edge. The same problem arises with any of the patios with one side equal to 3 because the hexamer must lie on one side and the only way you can fill the remaining row is with 2 dimers placed end to end. All such patios are imperfect.

Patios of order 4 (i.e. n × 4)

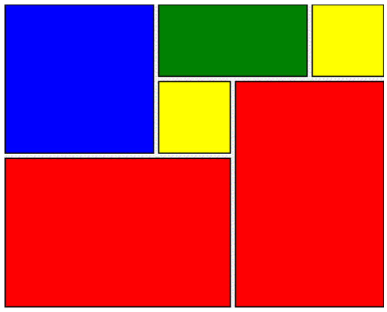
Things get a bit more interesting when we turn our attention to patios with the short dimension equal to 4. (I shall refer to the short dimension of a rectangular patio as the *order* of the patio.) There are quite a few pavings of the 5 × 4 patio with small defects but only three which are perfect namely:



Perfect 5 × 4 paving

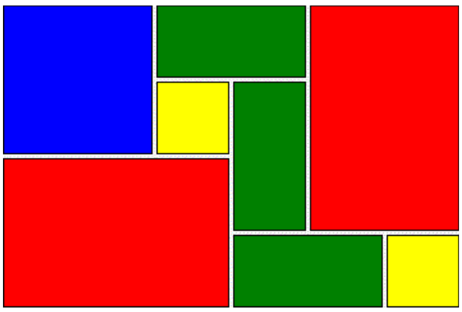


Perfect 5 × 4 paving



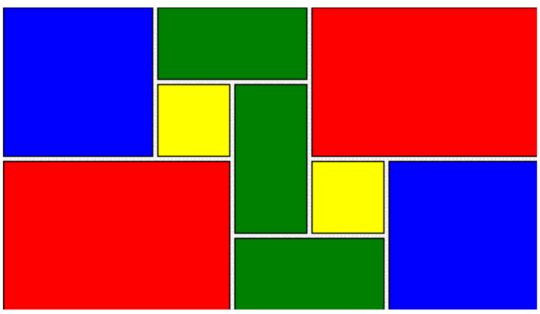
Perfect 5 × 4 paving

With the 6 × 4 patio, a lot more possibilities arise but in spite of this, I have only found one solution:

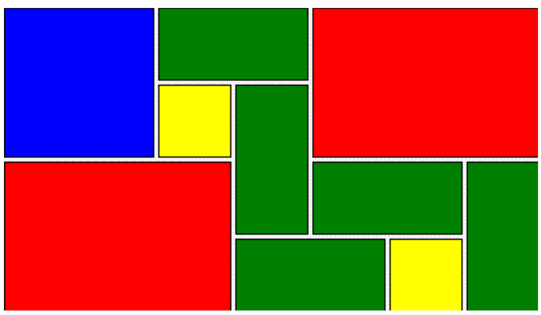


Perfect 6 × 4 paving

The 7 × 4 patio is big enough for it to contain two each of the larger slabs:

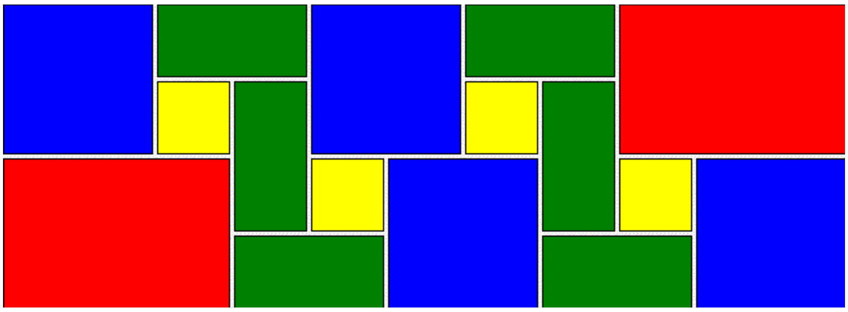


Perfect 7 × 4 paving with rotational symmetry



Perfect 7 × 4 paving

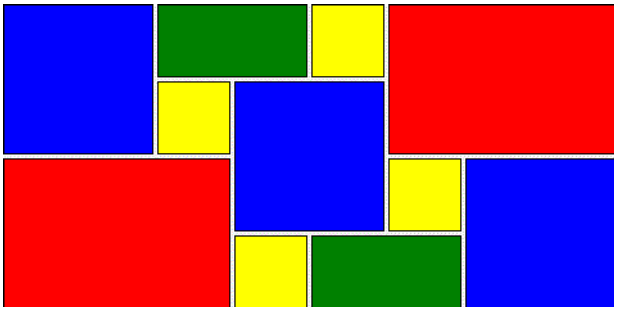
One interesting thing about the symmetric solution is the Z figure in the middle which has width 2. If you add two further staggered tetramers or 2 further hexamers, you can extend this solution by 4 or 5 units as many times as you like. For example this is a solution for an 11 × 4 patio:



Perfect 11 × 4 paving

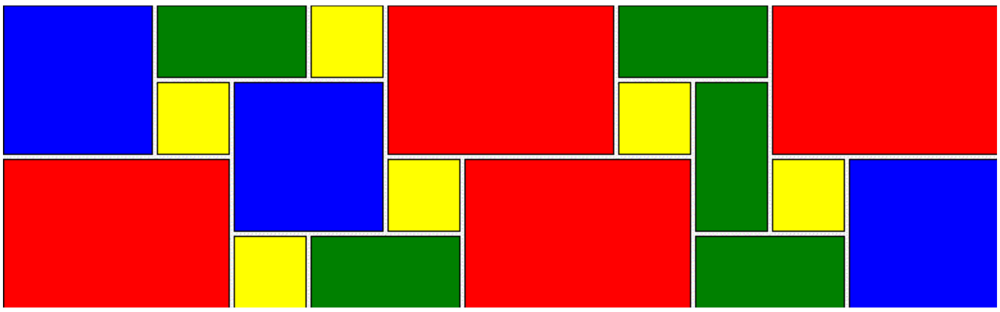
Because you can extend the patio by either 4 or 5 units, similar solutions exist for all unit 4 patios except 8, 9, 10, 13, 14 and 18.

The 8 × 4 patio has a perfect solution of a slightly different type, namely:



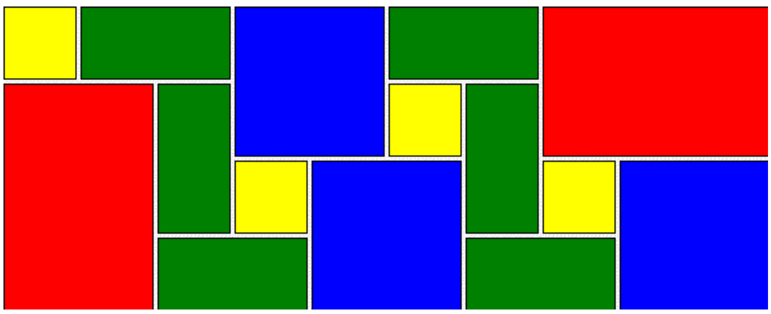
Perfect 8 × 4 paving

This time, the connecting Z piece has width 3 so we can extend this solution by either 5 or 6 units. This enables us to generate a solution to the 13 × 4 patio:



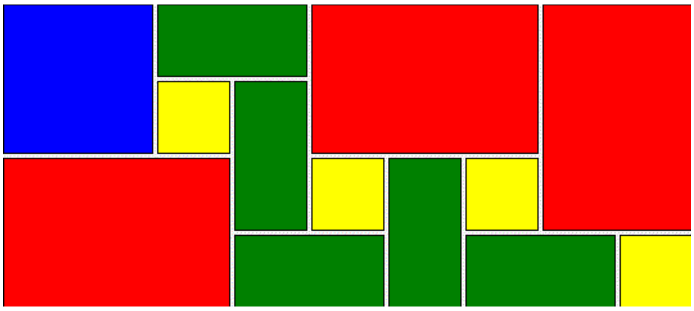
Perfect 13 × 4 paving

and by juggling the components,. The 14 × 4 and 18 × 4 patios as well. This just leaves the 9× 4 and 10× 4 patios to solve. Here is a solution to the latter:



Perfect 10 × 4 paving

This just leaves the 9 × 4 patio to solve. I cannot find a symmetrical solution but this asymmetrical one will do:



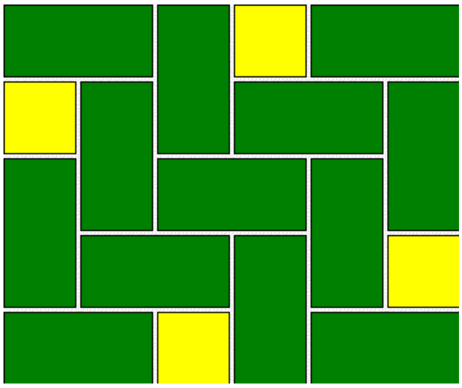
Perfect 9 × 4 paving

but it has rather too many hexamers for my liking.

It is one thing to prove that perfect pavings exist for all *n* × 4 patios but there is still a lot of fun trying to find all the solutions that exist for a given patio but I will leave that task for another day. First lets see what we can say about patios with a short side of 5.

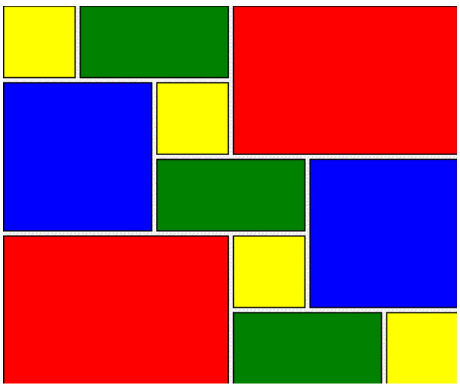
Higher order patios

There is a very elegant 5 × 5 Tatami tiling but it does not translate directly into a patio paving.

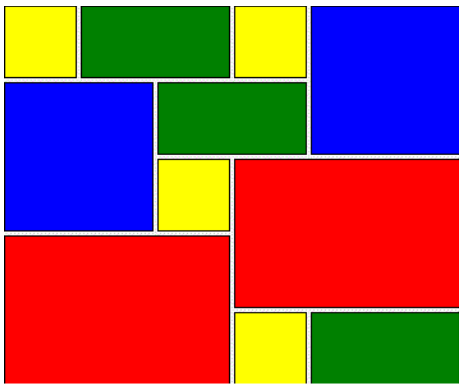


5 × 5 Tatami tiling

There are, however, some equally elegant perfect patio pavings:

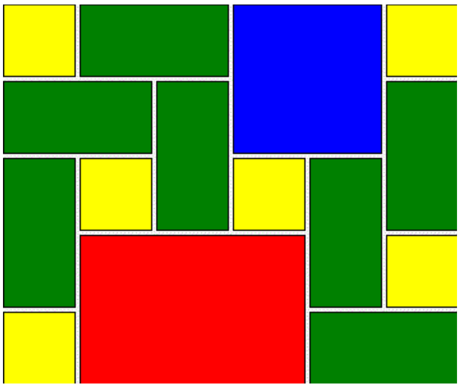


Perfect 5 × 5 paving

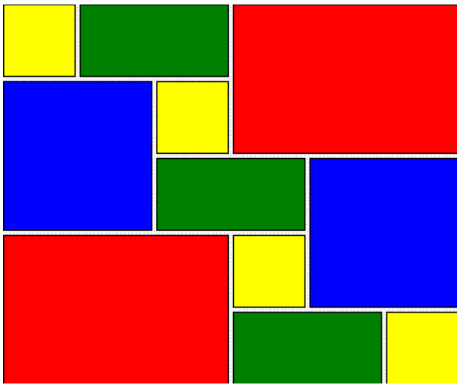


Perfect 5 × 5 paving

Moving on to 6 × 5 pavings, of the possible solutions here are two examples:



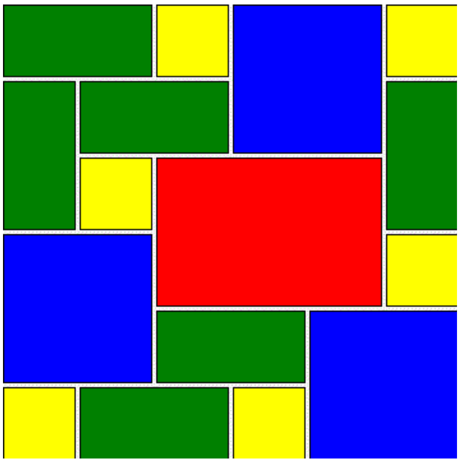
Pretty perfect 6 × 5 paving



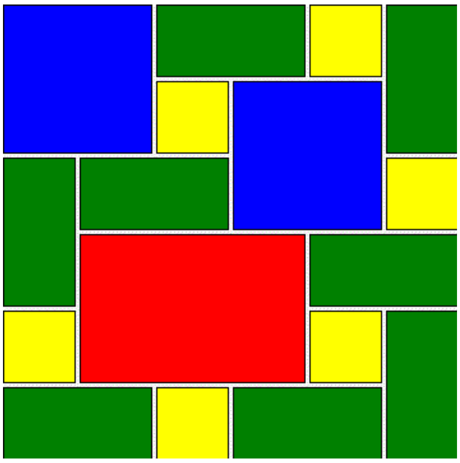
Perfect 6 × 5 paving

Of the two I think I prefer the first, partly because of its randomness but also because it has, to me, a better balance of large and small pavers. This suggests to me a further condition that we might place on large patios – namely that two large pavers (i.e. tetramers and hexamers) should have no contact at all. I shall call these *pretty perfect patio pavings*.

There are lots of perfect 6 × 6 pavings but I cannot find any symmetrical solutions and none of them are 'pretty'. These are my favourite 6 × 6 solutions.



Perfect 6 × 6 paving



Perfect 6 × 6 paving

Neither of them are pretty (according to my mathematical definition) but the second of the examples above is, to my mind at any rate, a particularly pleasing perfect patio paving!

When we consider even larger rectangles, we can begin to insist on the 'pretty' condition. Indeed, we might consider adding a further condition – that the tetramers and hexamers should not lie along the edge of a rectangle. We might call these patios *particularly pretty perfect patio pavings*. I do not know if any such solutions exist.

It is also possible that there exist certain rectangles which do not have any perfect solutions at all, pretty or otherwise.

Do algorithmic solutions such as I have devised for patios of order 4 exist for other orders? If so, which?

The search for answers to these questions promises much further amusement but these deliberations have suggested a problem of even greater interest, namely: is it possible to tile the plane with a pretty perfect patio paving?

Pretty perfect plane patio pavings

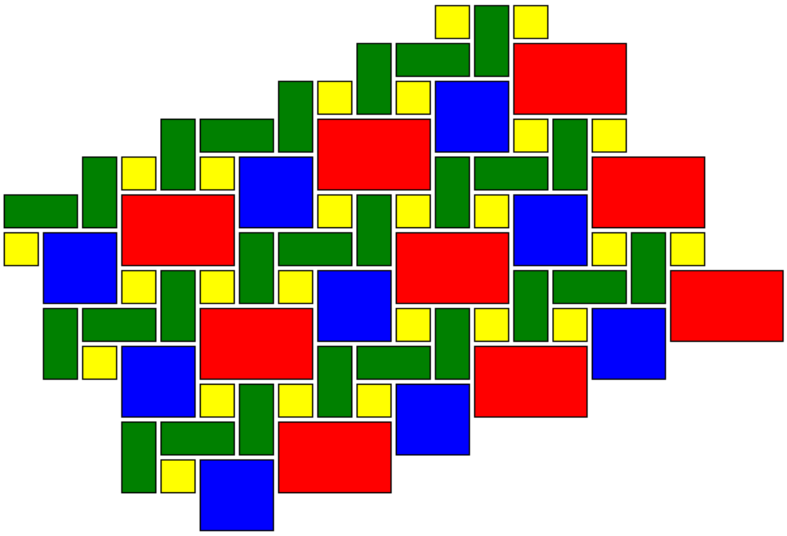
Let us review the conditions needed for a pretty perfect plane patio paving.

For a paving to qualify as perfect it must satisfy three conditions:

1. It must not contain any + crossings
2. It must not contain any common edges (except between a dimer and a minomer)
3. It must not have any cracks

For it to qualify as a patio paving, it must contain one or more of each of monomers, dimers, tetramers and hexamers and for it to qualify as a pretty patio paving the tetramers and hexamers must not touch.

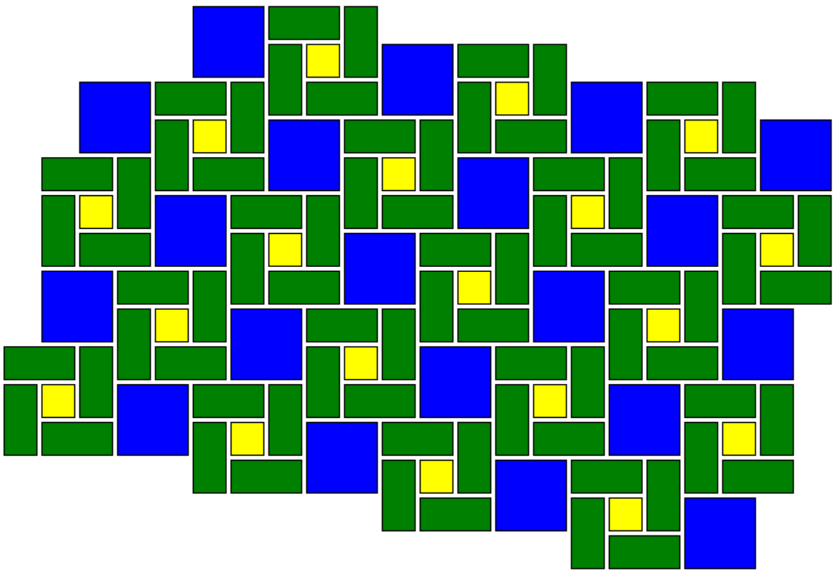
It turns out that these five condition have just the right mix to generate a problem of absorbing fascination. It is fairly easy to find perfect plane patio pavings (i.e. which do not qualify as being pretty on account of their touching tetramers and hexamers). Here is an example:



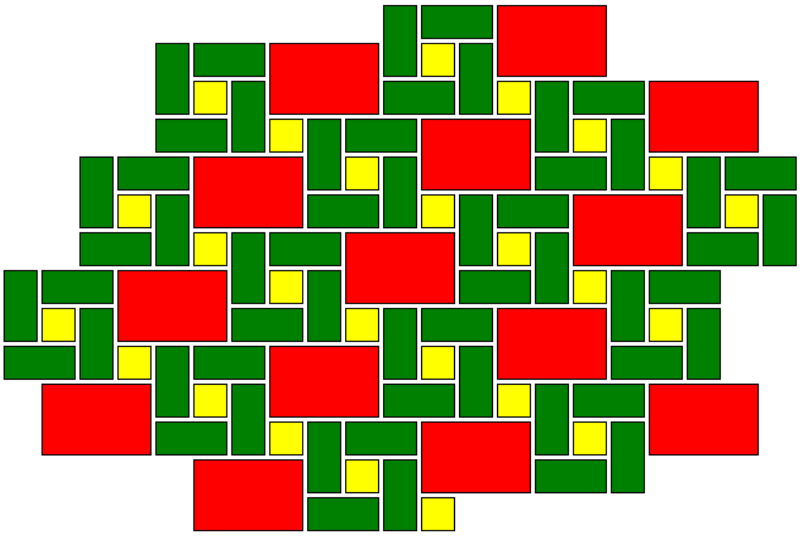
Perfect plane patio paving 3-3-1-1

(The numerical designation refers to the number of monomers, dimers, tetramers and hexamers in each unit cell.)

It is also quite easy to find pretty perfect pavings (which do not qualify as being patio pavings on account of the lack of one of the large pavers) e.g.

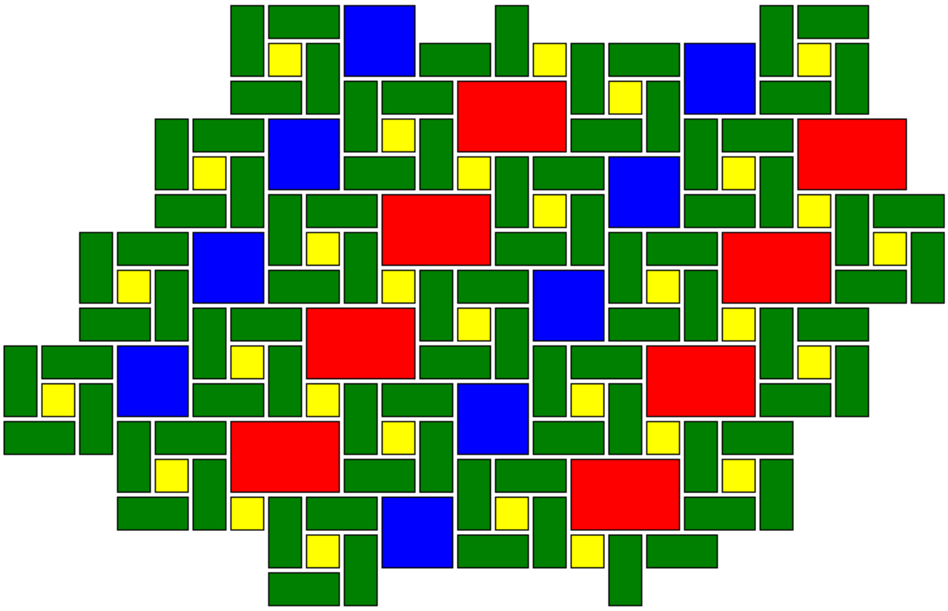


Pretty perfect plane paving 1-4-1-0



and *Pretty perfect plane paving 2-4-0-1*

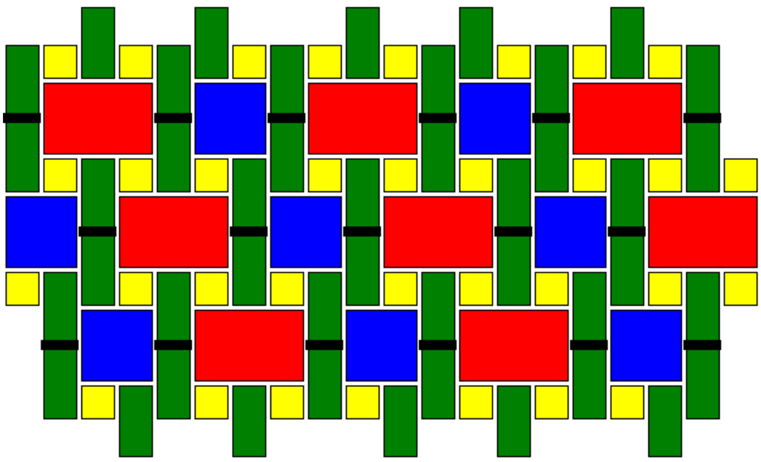
Both of these employ the same kind of construction. It is therefore possible to combine alternate strips from each one to achieve what we desire, namely a pretty perfect plane patio paving:



Pretty perfect plane patio paving 3-8-1-1

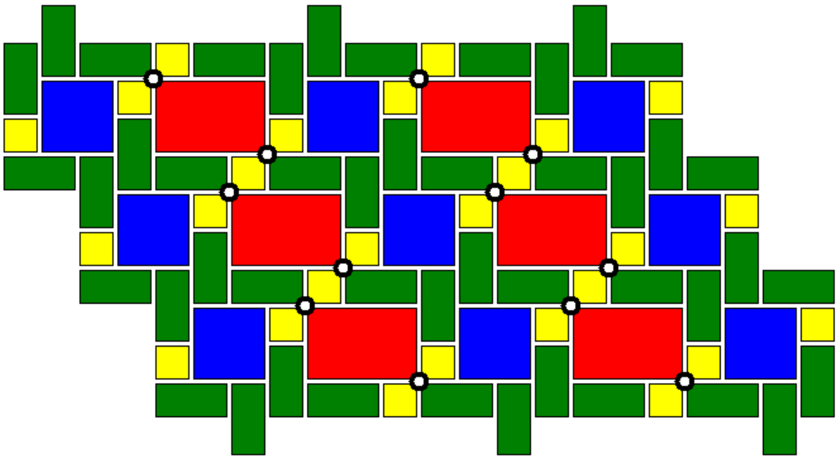
The question now arises, is this solution unique?

Whenever you try other possible combinations, imperfection always seem to arise. e.g.



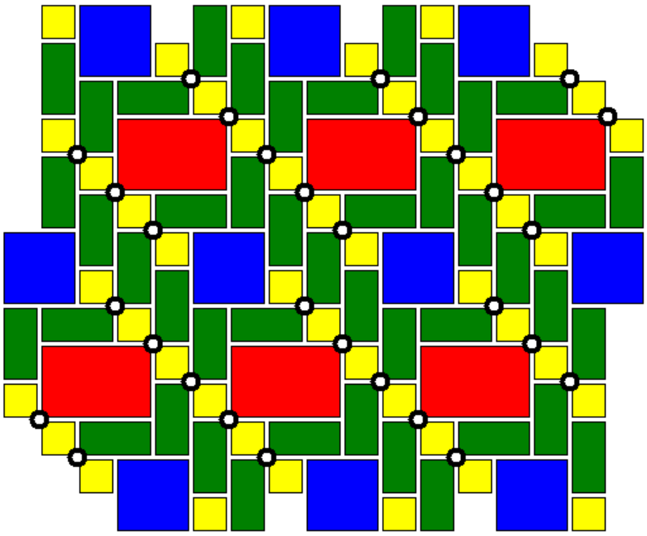
Pretty plane patio paving 3-4-1-1

which has two butting dimers per unit cell. You can modify this one as follows but you only replace one type of flaw with another:



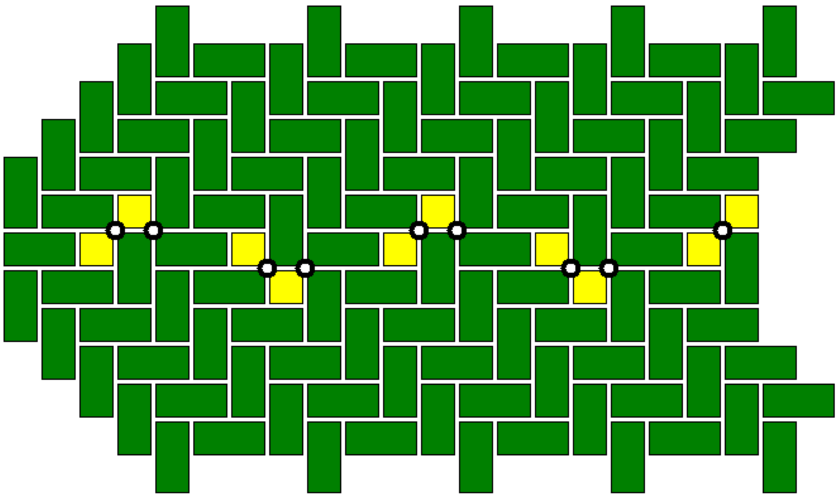
Pretty plane patio paving 3-4-1-1

This example has 5 + crossings per unit cell:

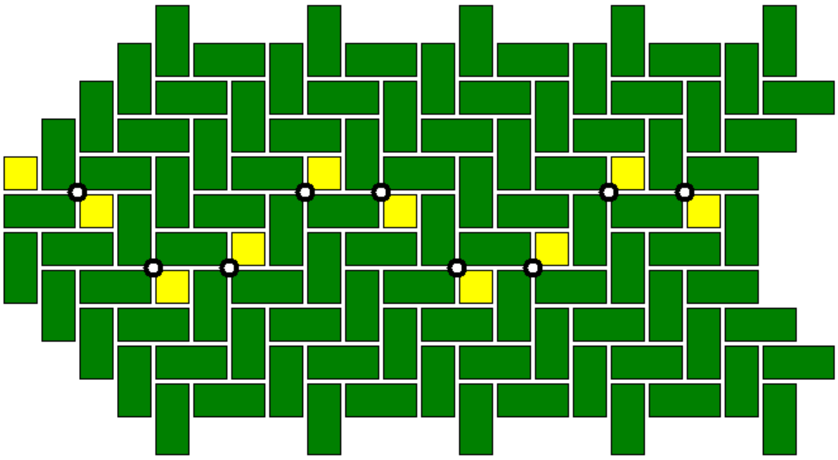


Pretty plane patio paving 6-7-1-1

All these examples are based on the idea of a unit cell but this is not a necessary condition. Many interesting pavings can be based on the intersection of two 'herringbone' patterns. In fact there are two ways in which two oppositely handed herringbone patterns can meet, namely:



Seam between herringbone patterns with odd parity which had odd parity (note the centre line of tiles) and

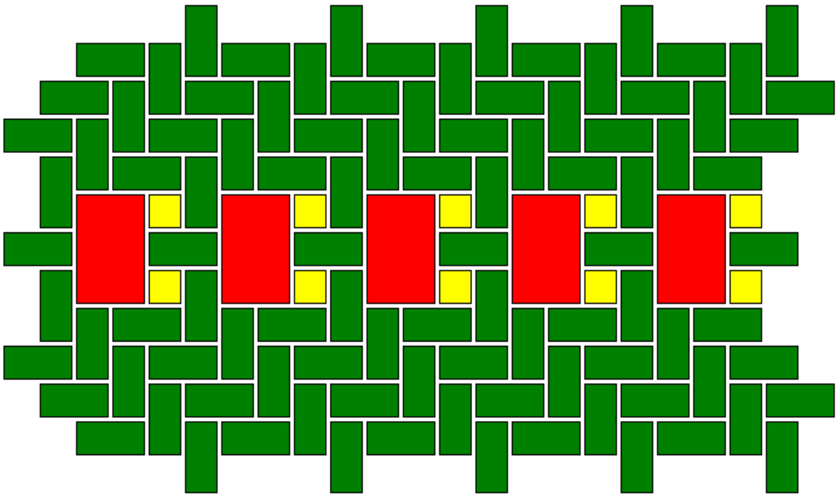


Seam between herringbone patterns with even parity

which is even.

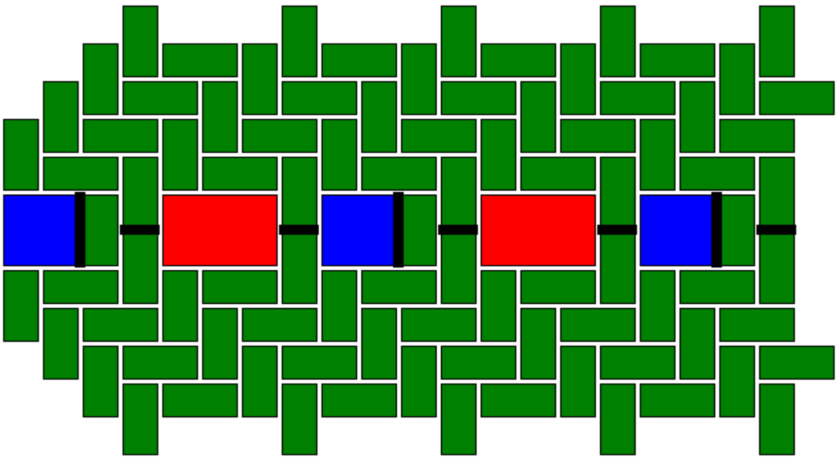
Both have numerous + crossings along the join.

It is, however, possible to eliminate these in the odd example above by using a line of vertical hexamers as follows:



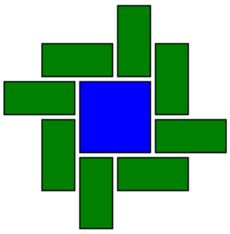
Odd parity seam joined with hexamers

Unfortunately, any attempt to join the patterns in the even case using either hexamers or tetramers involves butting pavers:



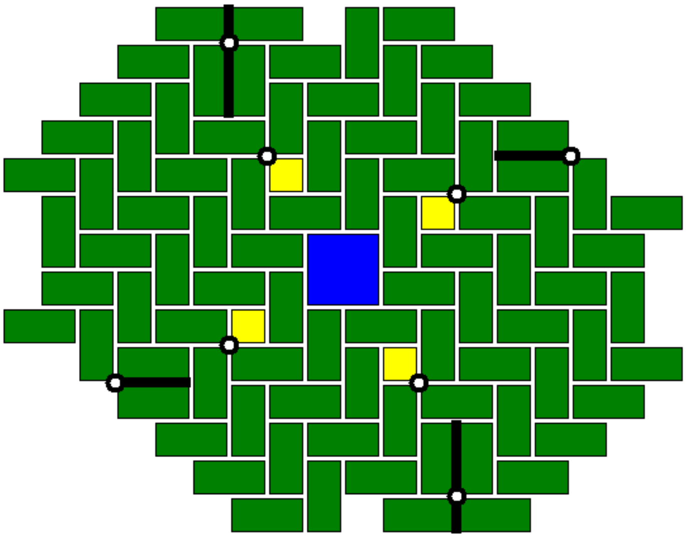
Attempts to join an even parity seam fail

The problem lies essentially with the tetramer. Because of its rotational symmetry, there is only one way (not counting its reflection) in which it can be completely surrounded by dimers without contravening one of the three conditions for a perfect paving, namely:.

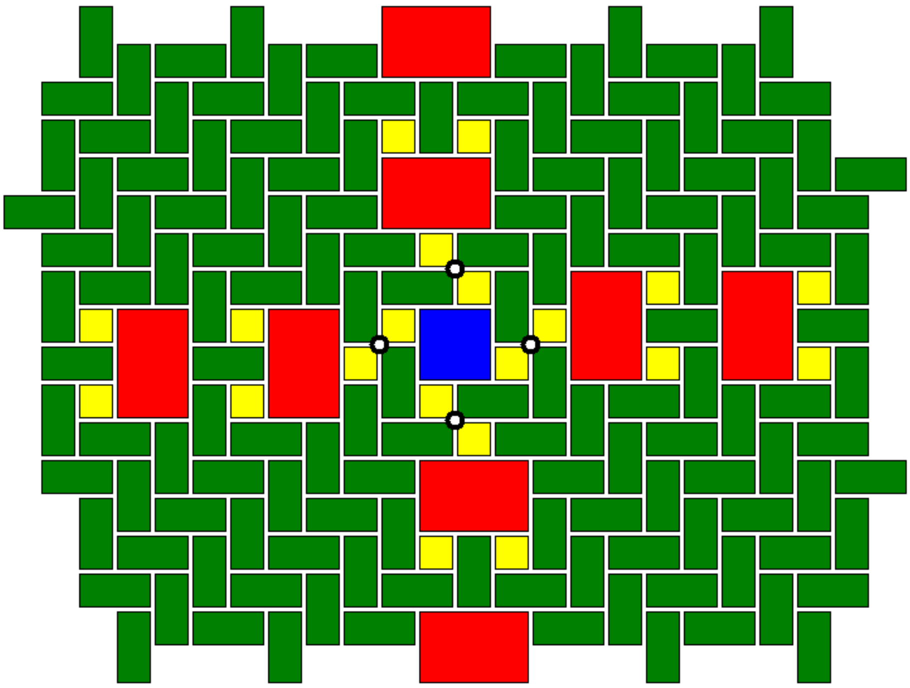


which I call the catherine-wheel.

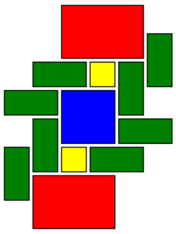
But if you try to extend the catherine-wheel by adding more dimers this leads to inevitable trouble where the resulting contrary herringbone patterns meet.



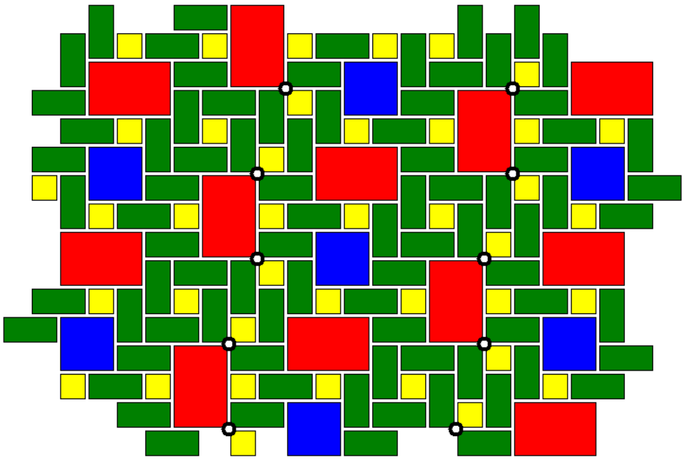
Now as it happens, the herringbone joins are all odd so it is possible to stitch them together using hexamers as shown on the previous page. The result is as follows:



There are other ways of getting over the problem but they all seem to run into trouble eventually. If we start with the following configuration:

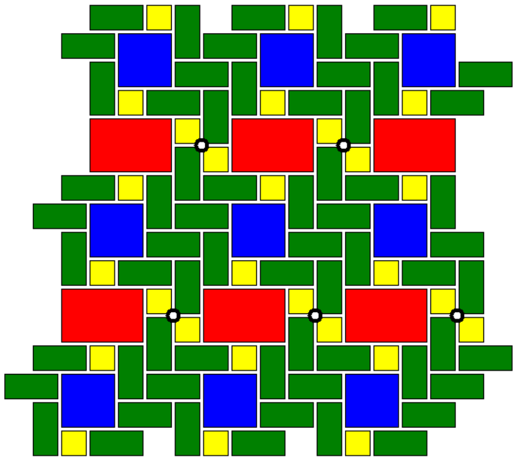


we can generate at least two almost perfect pretty plane patio pavings, namely:



Almost perfect plane paving 7-14-1-2

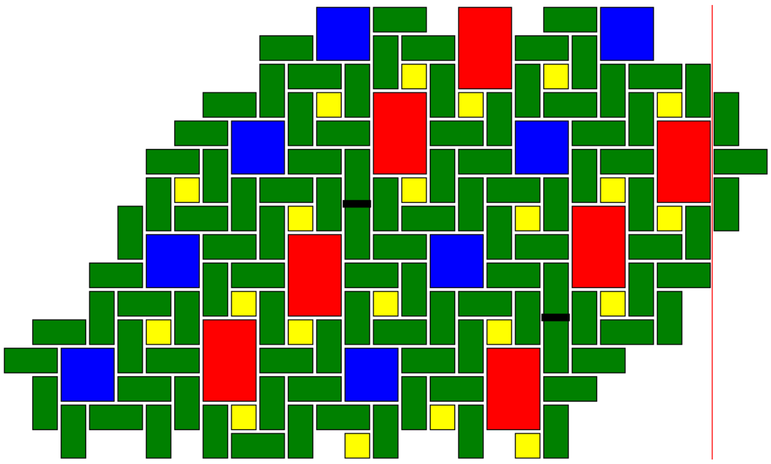
and



Almost perfect plane paving 4-8-1-1

the second of which only has 1 flaw per unit cell though the first is unquestionably the prettier of the two on account of its alternating hexamers.

I have discovered another almost perfect paving which only has one flaw in it per unit cell – a pair of butting dimers:



Almost perfect plane paving 6-27-2-2

but the question of whether the perfect pretty plane patio paving discovered earlier is unique still remains unresolved.