Black Body Radiation

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Abstract: The historical development of Planck's law from Rayleigh-Jeans law and Wien's Distribution law is described and a simple analogy is used to explain why the quantization of energy leads to the correct expression and the first estimate of Planck's constant.

Introduction

It is a matter of common experience that when you heat up an object like a poker in a fire, it first glows a dull red then, as the temperature rises, it does two things; first it emits a lot more heat and second it changes colour to yellow and even white. In 1893 the young German scientist Wilhelm Wein deduced from thermodynamic principles that the wavelength at which maximum energy was released should be inversely proportional to temperature – i.e.

$$\lambda_{max} = b/T \tag{1}$$

and experiments showed that the constant in this formula (known as Wien's *displacement* law) was equal to 0.0029 K m (i.e. at a temperature of 1000K a red hot poker emits its maximum energy at a wavelength of 2.9×10^{-6} m which is in the infra red region of the electromagnetic spectrum) but his theory gave no clue as to why the constant should be this value, nor did it seem to bear any relation to other known physical constants such as Boltzman's constant or the speed of light. The unit seemed odd too – Kelvin × metres. What did it mean?

Further experiments revealed that the graph of energy against wavelength at any given temperature had a well-defined bell-shaped curve shown in *fig. 1* but the first attempt to derive anything like this curve from first principles was a disaster.

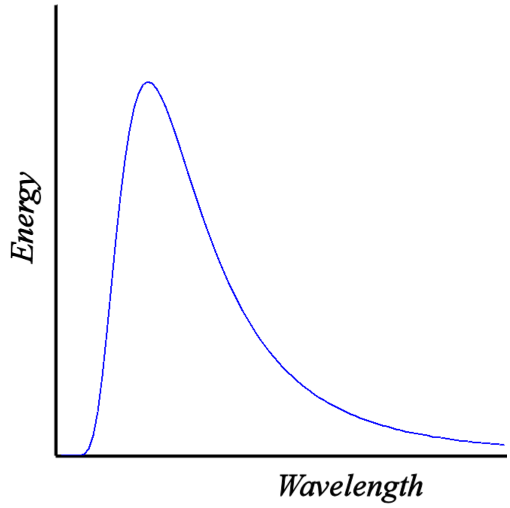

fig. 1

Rayleigh-Jeans Law

By considering what frequencies of radiation could exist inside a closed cavity at a temperature *T*, Rayleigh and Jeans concluded that the energy density per unit wavelength E_λ was inversely proportional to the fourth power of the wavelength.

$$E_\lambda = \frac{A}{\lambda^4} \qquad (2)$$

and, unlike Wien's displacement law, Rayleigh and Jeans' theory predicted the value of the constant $A = 2 \times c \times k \times \mathrm{T}$ where c is the velocity of light and k is Boltzman's constant.

What this means is that as the wavelength increases, the energy output by a black body decreases drastically or to put it even more simply – red hot pokers do not produce much in the way of radio waves, a fact well born out by experience!

If we plot Rayleigh and Jeans' prediction on the same graph as the experimental curve, this is what we get *(fig. 2)*:

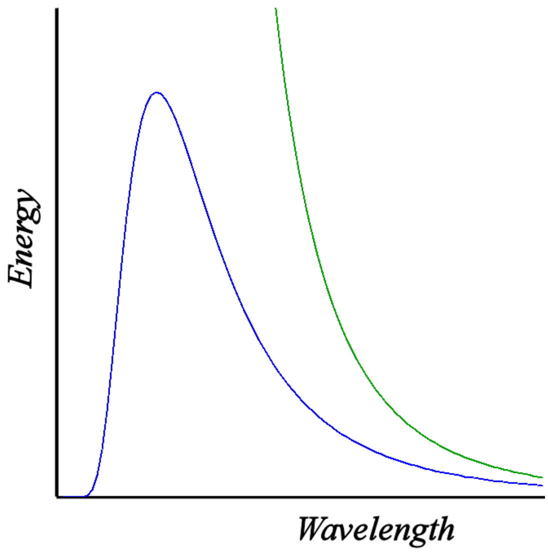

fig. 2

Although it doesn't look much like it on this diagram, at very long wavelengths the fit is, in fact, extremely good but it is immediately obvious that at all moderate wavelengths (where the body is emitting most of its radiation) the prediction is badly wrong and – worse still – the curve is completely the wrong shape! At shorter and shorter wavelengths the curve goes on rising without limit. The reason for this is basically that the shorter the wavelength, the more different wavelengths you can pack in in a given length.

To see why this is so, consider a simpler example. Suppose you twang a guitar string 60 cm long in the middle. The most prominent vibration will be the fundamental. But if you twang it near one end, the quality of the note produced will change because you will excite many of the higher harmonics. Suppose you twang it an infinitesimal distance from the bridge (impossible in practice, I know); theory suggests that *all possible modes of vibration* will be excited equally. Now let us consider how many modes of vibration exist which have wavelengths between 9.5 cm and 10.5 cm. There is only one, namely the 6^{th} harmonic which has a wavelength of 10 cm. The next shortest is 60/7 = 8.6 cm and the next longest is 60/5 = 12 cm. How many are there between 2.5 cm and 3.5 cm? A few quick calculations will reveal that at least 6 harmonics have wavelengths which lie in this range: numbers 17, 18, 19, 20, 21 and 22. What about the range 0.5 to 1.5 cm? (The answer is a whopping 79) It is obvious that the shorter the wavelength, the more modes there are within a given range.

Now it is a fundamental principle of classical thermodynamics called the *equipartition of energy* that if a system has many different ways of absorbing energy then, on average, every mode will share the same proportion of that energy. For example, a nitrogen molecule has 5 ways of storing energy – three translational ones and two rotational ones – which is why the specific heat capacity of nitrogen is 5/2 **R**, not 3/2 **R** as with monatomic gases such as helium which only have 3 degrees of freedom. In a cavity containing radiation there are an infinite number of possible modes of vibration – most of them with very short wavelength, and if you were to let this radiation out of the cavity a stream of X rays and gamma rays would emerge. The utter failure of the classical theory of thermodynamics to explain such an apparently simple phenomena as the colour of a red hot poker was dubbed the 'ultraviolet catastrophe' at the time. The 'gamma ray catastrophe' would have been a better phrase but at the end of the 19^{th} century, gamma rays were not known to be electromagnetic in nature.

Wien's Distribution Law

Wilhelm Wien continued to work at the problem an a few years later came up with an empirical formula which fitted the data very well at short wavelengths but fared poorly at long ones where the Rayleigh-Jeans law seemed to work much better. His formula was as follows:

$$E_\lambda = \frac{A}{e^{B/\lambda}\,\lambda^5} \qquad (3)$$

where *A* and *B* are constants which had to be 'tweaked' to give the right answers and pretty good it was too. *Fig. 3* shows a comparison. (The experimental data is the blue curve and Wien's approximation is the red one.)

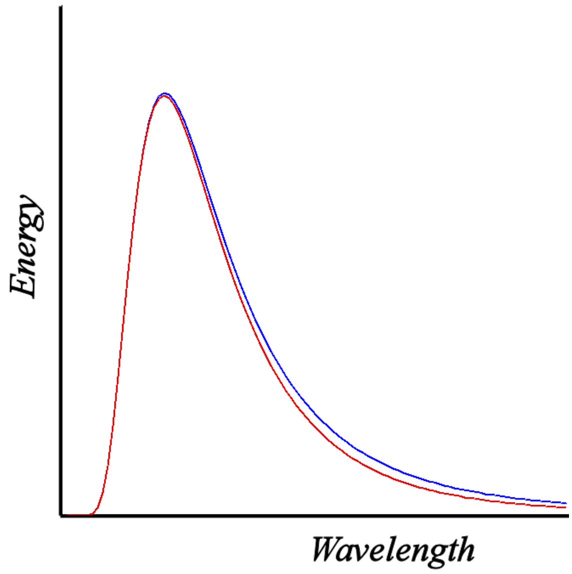

fig. 3

The fit at short wavelengths is extremely good and the overall shape is right but Wien's formula consistently underestimated the energy radiated at long wavelengths. Annoying though this discrepancy was, the real problem with Wien's formula was that, without any theoretical justification, there was no way of predicting the values of *A* and *B*.

Planck's law

So now we had an excellent theory which just didn't fit the facts and an excellent formula for which there was no theoretical justification whatsoever. It was at this point, during a lunch in October 1898 with his friend Heinrich Rubens who had recently built an instrument which could measure the radiation from a black body with extreme accuracy, that Max Planck, professor of theoretical physics at Berlin university, decided to have a go at finding a formula that would fit the data at all points. This is what he had to go on:

$$E_\lambda = Ae^{-B/\lambda}/\lambda^5 \qquad \text{<...... ? ......>} \qquad = C/\lambda^4$$

Short wavelengths Long wavelengths

That very evening the solution came to him. What if, instead of using the expression $e^{-B/\lambda}$ we were to use the expression $(e^{B/\lambda}-1)^{-1}$ instead? - the point being that whenever λ is small, $e^{B/\lambda} \gg 1$ and the expression boils down to $e^{-B/\lambda}$ as required.

Here is the formula which Planck proposed:

$$\boxed{E_\lambda = \frac{A}{(e^{B/\lambda} - 1)\lambda^5}} \tag{4}$$

But what if λ is large? Does it boil down to the Rayleigh-Jeans formula? Incredibly. It does.

You see, if λ is large, $B/\lambda \ll 1$ and we can use the Taylor expansion of e^x, namely

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

So, throwing away all the higher order terms we find that

$$e^{B/\lambda} \approx (1 + B/\lambda) \tag{5}$$

Substituting in equation (12) we get:

$$E_\lambda \approx \frac{A}{(1 + B/\lambda - 1)\lambda^5} = \frac{A/B}{\lambda^4} \tag{6}$$

Comparing notes with his friend's experimental results the next day, Planck was stunned by the accuracy of his new formula. But what did it mean? Where was the theoretical justification for such a bizarre expression? There was nothing at all in classical physics which seemed to give a remote clue as to where to start looking for an explanation.

To Planck the problem seemed to be with the fundamental principle of equipartition of energy. What if there was some mechanism that prevented the short wavelength, high frequency modes of oscillation from being excited? Was it, perhaps, that short wavelength modes 'cost' more? Perhaps the available modes of oscillation are like a series of bells whose frequencies form a harmonic series with frequencies f, $2f$, $3f$, $4f$ etc. etc. but it costs £1 to ring the lowest bell, £2 to ring the $2f$ bell, £3 to ring the $3f$ bell etc. etc. How many bells can you ring with £100?

fig. 4

Well, you could ring the biggest bell 100 times or you could ring all the bells from f to $13f$ once because 1+2+3+4+5+6+7+8+9+10+11+12+13 = 91. Alternatively you could choose to ring just the 4 bells $10f$ $20f$ $30f$ and $40f$ if you wanted. Now suppose that you are given £100 and asked to spend the *same amount of money* ringing each bell. Obviously you can't ring any of the bells over $100f$ because you haven't got enough money to ring them even once. A moments thought will serve to convince you that it is no good spending money on any of the expensive bells above 50*f* because that won't leave you with enough money to share out amongst all the other bells. In fact, you might conclude that there is *no* way you can meet the requirement. But suppose I insist that you do the best you can. You might decide to ring 10 bells and spend approximately £10 on each like this:

$$10 \times £1,\ 5 \times £2,\ 4 \times £3,\ 3 \times £4,\ 2 \times £5,\ 2 \times £6,\ 2 \times £7,\ 1 \times £8,\ 1 \times £9 \text{ and } 1 \times £10$$

Now lets consider what *wavelengths* we have produced. Suppose for the sake of argument that the lowest bell produces a sound whose wavelength is 100 cm. We can rewrite the above list (in cm) as follows:

$$10 \times 100,\ 5 \times 50,\ 4 \times 33,\ 3 \times 25,\ 2 \times 20,\ 2 \times 17,\ 2 \times 14,\ 1 \times 12,\ 1 \times 11 \text{ and } 1 \times 10$$

Now lets plot a graph of money spent against wavelength. We shall divide the wavelengths into 10 groups of 10 cm each i.e. 0-9cm, 10-19cm, 20-29cm etc. and count up how much money was spent in each group. The results are as follows:

0-9	0
10-19	£10 + £9 + £8 + 2 × £7 + 2 × £6 = £53
20-29	2 × £5 + 3 × £4 = £22
39-39	3 × £3 = £9
40-49	0
50-59	5 × £2 = £10
60-69	0
70=79	0
80-89	0
90-99	0
100-109	10 × £1 = £10

And the graph looks like this *(fig. 5)*:

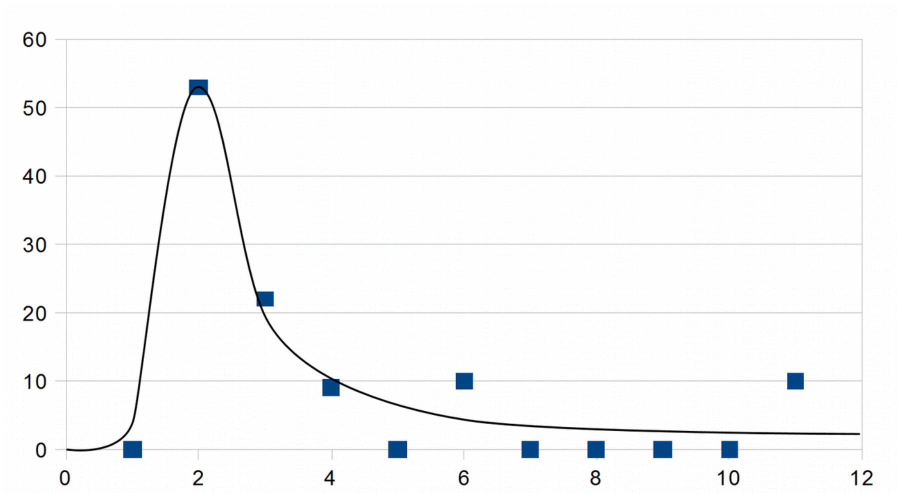

fig. 5

which is beginning to resemble the curve we are after.

I have no idea whether or not some such analogy occurred to Planck but for whatever the reason he began to conceive of the idea that the high frequency modes of oscillation needed more energy to excite them than the low ones according to the simple formula

$$\Delta E = h\nu \tag{7}$$

where h is some constant.

The reason, therefore, why black bodies do not emit large quantities of gamma radiation is the same as the reason why you can't ring the bells with frequencies above 100f. There just isn't enough energy around to get these modes excited. And the reason why hot bodies don't emit very much in the way of radio waves is that there aren't nearly as many possible modes of vibration at these low frequencies.

To cut a long story short, when Planck turned this simple idea into mathematical form, what did he find? He found that the radiation law would have to be precisely the law which he had proposed two years earlier. What is more, he now had a theory which enabled him to replace all those arbitrary empirical constants with well known physical constants such as Boltzmann's constant k, the speed of light c and his new constant h. This is the formula he derived:

$$\boxed{E_\lambda = \frac{2hc^2}{(e^{hc/kT\lambda} - 1)\lambda^5}} \tag{8}$$

If you look back at equation (4) you will see that the two constant there, A and B, have the following values:

$$A = 2hc^2 \quad \text{and} \quad B = hc/kT \tag{9}$$

This means that the constant in the Rayleigh-Jeans formula (equation (6)) is

$$A/B = 2ckT \tag{10}$$

which is exactly what Rayleigh and Jeans had discovered.

What is more, by differentiating equation (8) it is possible to determine the value of λ at which E_λ is maximum. It turns out that solving for λ is a bit more difficult than it looks. It is a lot easier to differentiate Wiens' approximation (equation (3)) and a glance at f*ig 3* will convince you that the maxima are in virtually the same place. The upshot is that:

$$\lambda_{\text{max}} = B/5 = hc/5kT \tag{11}$$

But this means that, going right back to equation (1) where we recall that

$$hc/5k = 0.0029 \tag{12}$$

Putting in the well-known values of c and k

$$h = \frac{0.0029 \times 5k}{c} = \frac{0.0029 \times 5 \times 1.4 \times 10^{-23}}{3 \times 10^8} = 6.76 \times 10^{-34}\,\text{J s} \tag{13}$$

Planck described his new theory to the German Physical Society on the 14th of December 1900. He was warmly congratulated on his success but nobody in the room, least of all Planck himself believed that the quantization of energy implicit in equation (7) was real. To him it was simply a curious mathematical device which got the right answers. Indeed, in the year 1900 many scientists including Planck were not even sure that atoms existed, let alone quanta. Little did they know what was in store for them.

Five years later, a diminutive clerk in the patent office in Bern was not only to prove that atoms existed (by explaining Brownian motion) and that quanta existed (by explaining the photoelectric effect) but that we were wrong about just about everything else which existed as well. But that is another story.